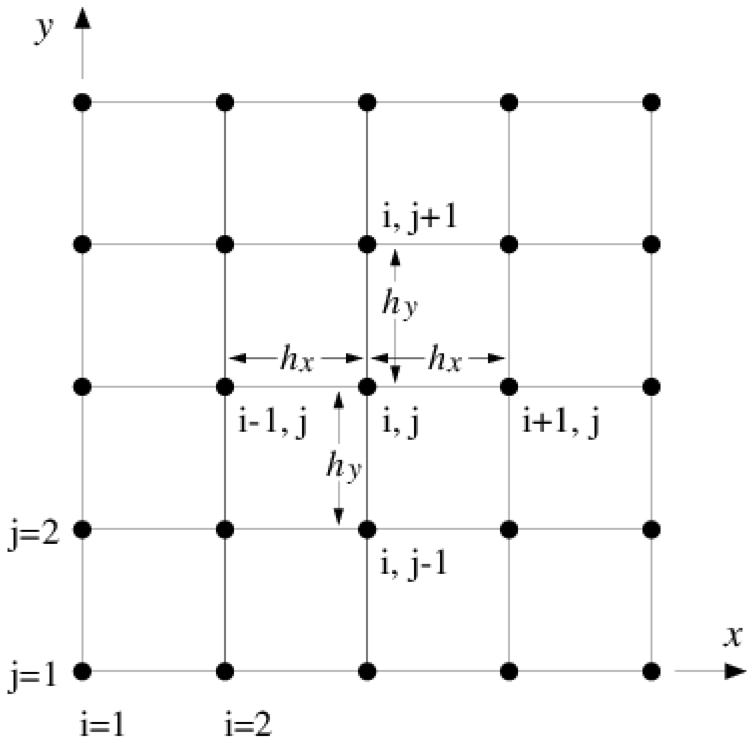
Numerical Viscosity and Diffusion in Finite Difference Eulerian Codes

Simon Candelaresi, Dominika Zieba



What is Numerical Diffusion?

Everyone is talking about it,
but no one knows what is really is.

Numerical Experiments

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On the Measurements of Numerical Viscosity and Resistivity in Eulerian MHD Codes

Tomasz Rembiasz[1,2], Martin Obergaulinger[1], Pablo Cerdá-Durán[1], Miguel-Ángel Aloy[1], and Ewald Müller[2]
[1] Departamento de Astronomía y Astrofísica, Universidad de Valencia, C/Dr. Moliner 50, E-46100 Burjassot, Spain; tomasz.rembiasz@uv.es
[2] Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, D-85748 Garching, Germany

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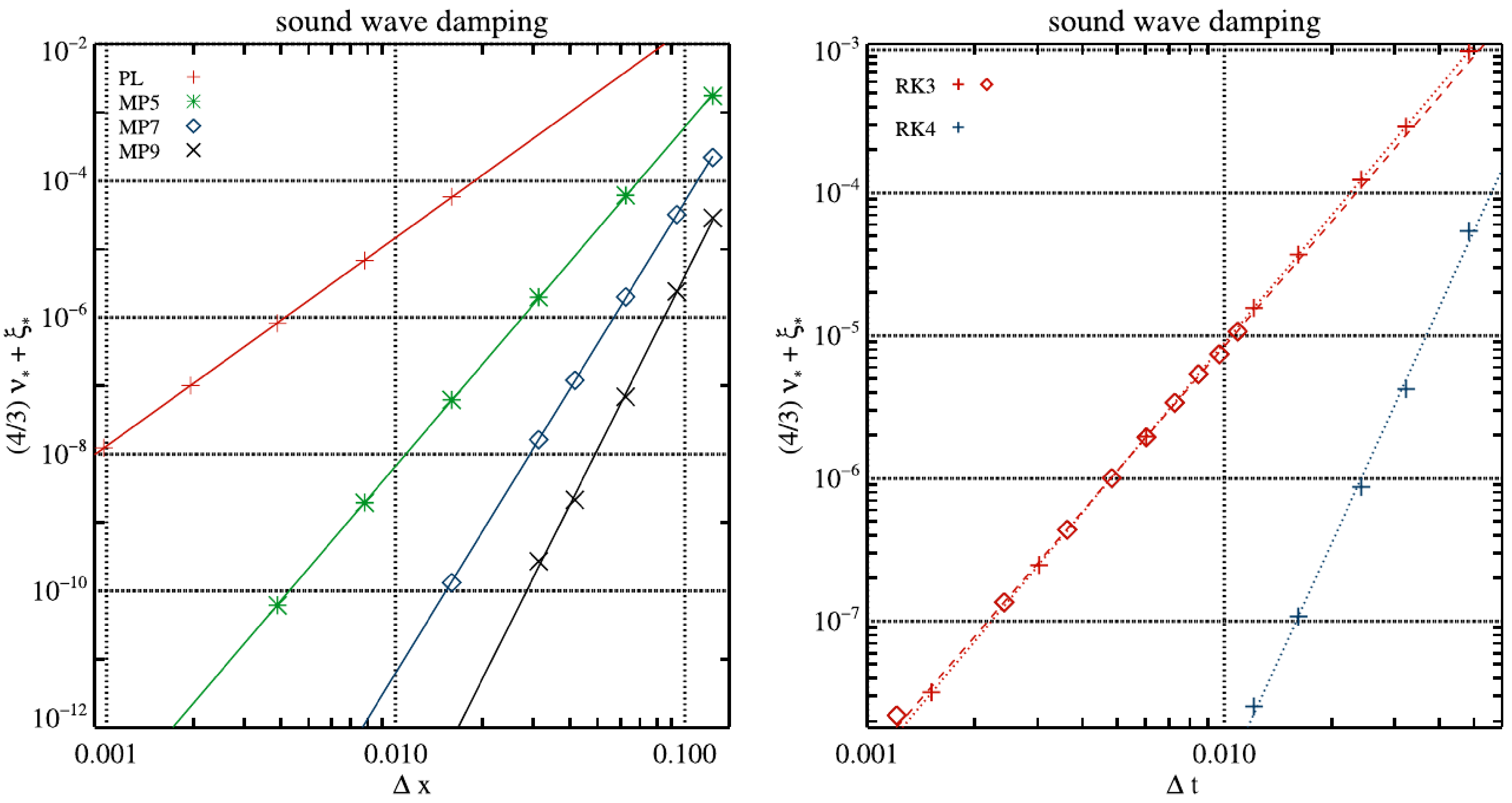
Table 1
Wave Damping Simulations I

Series	Wave	Reco	Riemann	Time	CFL	Resolution	$\mathfrak{N}^{\Delta x}_{\rm tot}$	r	$\mathfrak{N}^{\Delta t}_{\rm tot}$	q
#S1	sound	PL	HLL	RK4	0.01	64...1028	14.3 ± 0.7	3.049 ± 0.009	...	...
#S2	sound	MP5	LF	RK4	0.01	8...256	42.9 ± 2.3	4.957 ± 0.013	...	...
#S3	sound	MP5	HLL	RK4	0.01	8...256	43.4 ± 2.5	4.961 ± 0.014	...	...
#S4	sound	MP5	HLLD	RK4	0.01	8...256	42.7 ± 2.2	4.956 ± 0.013	...	...
#S5	sound	MP7	HLL	RK4	0.01	8...64	302 ± 20	6.897 ± 0.021	...	...
#S6	sound	MP9	HLL	RK4	0.01	8...32	830 ± 340	8.42 ± 0.15	...	...
#S7	sound	MP9	HLL	RK3	0.5	8...256	...	...	1.492 ± 0.013	2.985 ± 0.002
#S8	sound	MP9	HLL	RK3	0.1...0.9	64	...	...	2.45 ± 0.17	2.95 ± 0.01
#S9	sound	MP9	HLL	RK4	0.5	8...32	...	...	71 ± 32	5.5 ± 0.2
#A1	Alfvén	MP5	LF	RK4	0.01	8...256	42 ± 3	4.95 ± 0.02	...	...
#A2	Alfvén	MP5	HLL	RK4	0.01	8...256	42.6 ± 2.1	4.96 ± 0.01	...	...
#A3	Alfvén	MP5	HLLD	RK4	0.01	8...256	42 ± 3	4.95 ± 0.02	...	...
#A4	Alfvén	MP7	HLL	RK4	0.01	8 ...128	44 ± 53	6.19 ± 0.03	...	...
#A5	Alfvén	MP9	HLL	RK4	0.01	8...64	1190 ± 190	8.57 ± 0.06	...	...
#A6	Alfvén	MP9	HLL	RK3	0.8	16...128	...	...	0.86 ± 0.08	2.949 ± 0.022
#A7	Alfvén	MP9	HLL	RK4	0.8	8...64	...	...	7.6 ± 2.5	5.18 ± 0.10
#A8	Alfvén	MP5	HLL	RK3	0.5	5...1024	...	...	...	...
#MS1	magnetosonic	MP5	HLL	RK4	0.01	8...128	40 ± 3	4.95 ± 0.02	...	...
#MS2	magnetosonic	MP7	HLL	RK4	0.01	8...64	288 ± 20	6.903 ± 0.023	...	...
#MS3	magnetosonic	MP9	HLL	RK4	0.01	8...32	1970 ± 160	8.82 ± 0.03	...	...
#MS4	magnetosonic	MP9	HLL	RK3	0.1...0.9	64	...	...	1.77 ± 0.06	2.977 ± 0.007
#MS5	magnetosonic	MP9	HLL	RK4	0.2...0.9	64	...	...	4.3 ± 0.8	4.834 ± 0.013

Wave Damping



Analytical Approach

Numerical Methods

University of Glasgow

Radostin Simitev

July 23, 2019

Local Truncation Error

discretized   exact $\mathcal{L}[\hat{u}] = 0$

Definition 7.4. *The quantity*

$$\tau_{(h)} = \mathcal{A}_{(h)}[\hat{u}] - \mathcal{L}[\hat{u}] = \mathcal{A}_{(h)}[\hat{u}] = A_{(h)}\hat{u} - F_{(h)}.$$

is called the local truncation error (local residual) of the numerical scheme $\mathcal{A}_{(h)}[] = 0$.

Example 7.5. *Find the local truncation error of the numerical scheme*

$$\mathcal{A}_{(h)}[u] = \frac{u_{k-1} - 2u_k + u_{k+1}}{h^2} - f_k = 0$$

for the solution of

$$u'' - f = 0.$$

This is solved exactly.

Solution. Now

$$\mathcal{A}_{(h)}[\hat{u}] = \frac{\hat{u}_{k-1} - 2\hat{u}_k + \hat{u}_{k+1}}{h^2} - f_k = (\hat{u}_k'' + O(h^2)) - f_k,$$

but

$$\mathcal{L}[\hat{u}] = \hat{u}_k'' - f_k = 0,$$

so using the definition directly

$$\tau_{(h)} = \mathcal{A}_{(h)}[\hat{u}] - \mathcal{L}[\hat{u}] = O(h^2).$$

Numerical Diffusion

PDEs: $\mathcal{L}[\hat{u}] = 0$

In the PencilCode do we have $\mathcal{A}_{(h)}[\hat{u}] = c\partial_{xx}\hat{u} + \ldots$?

What is c ?

Approach:
1. Discretize PDEs.
2. Apply method of lines to get set of coupled ODEs.
3. Construct the Runge-Kutta intermediate steps.
4. Eliminate off-center values using the Taylor expansion.
5. Eliminate intermediate time steps using time Taylor expansion.

$$f_{i\pm 1} = f_i \pm \mathrm{d}x f_i' + \frac{\mathrm{d}x^2}{2} f_i'' \pm \frac{\mathrm{d}x^3}{6} f_i''' + \ldots$$

$$\frac{f_{i+1} - f_{i-1}}{2\mathrm{d}x} = f_i' + \frac{\mathrm{d}x^2}{6} f_i''' + \ldots$$

Inviscid Navier-Stokes 3d

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - c_\mathrm{s}^2 \nabla \ln\left(\rho\right)$$

second order space
second order Runge-Kutta

$$\frac{\partial \ln\left(\rho\right)}{\partial t} = -\mathbf{u} \cdot \nabla \ln\left(\rho\right) - \nabla \cdot \mathbf{u}$$

Truncation errors with $\partial_{xx} u_x$:

$$-\frac{c_\mathrm{s}^2 \mathrm{d}t^2 \mathrm{d}x^2 \ln(\rho)_{xxx} u_x}{24} - \frac{c_\mathrm{s}^2 \mathrm{d}t^2 \mathrm{d}x^2 \ln(\rho)_x u_{x,xx}}{8} + \ldots$$

Similar for $\partial_{yy} u_x$ and $\partial_{zz} u_x$.

Proper diffusion terms: $\partial_t u_x = \partial_{xx} u_x + \partial_{yy} u_x + \partial_{zz} u_x$

Conclusions

- Numerical viscosity and diffusion can be calculated analytically.
- Need to find proper interpretation of the terms.
- Next: higher order space and time discretization, MHD.

simon.candelaresi@gmail.com