Lagrangian Relaxation of Magnetic Fields

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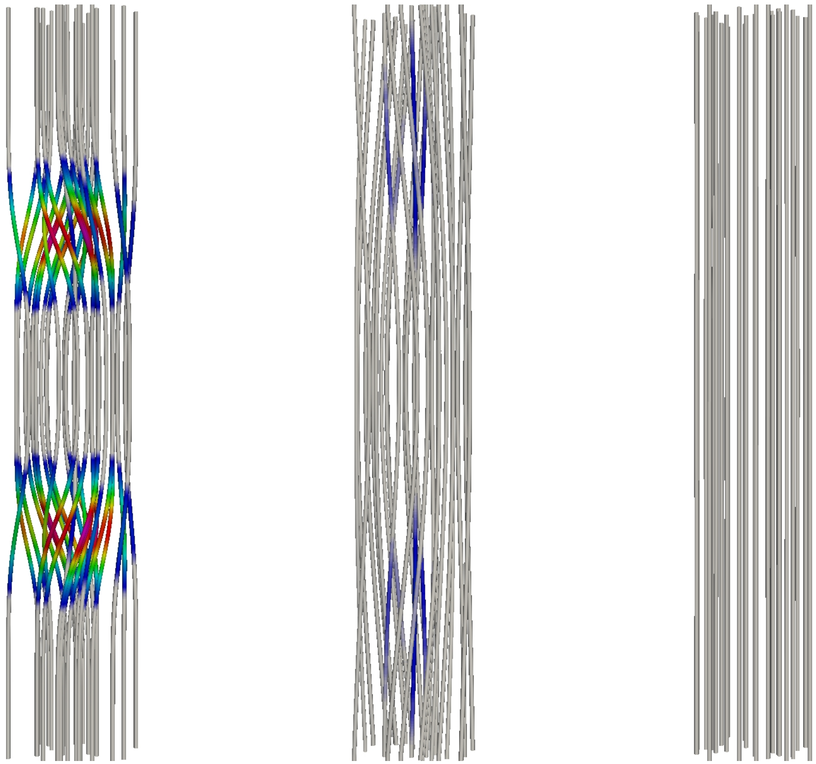


Force-Free Magnetic Fields

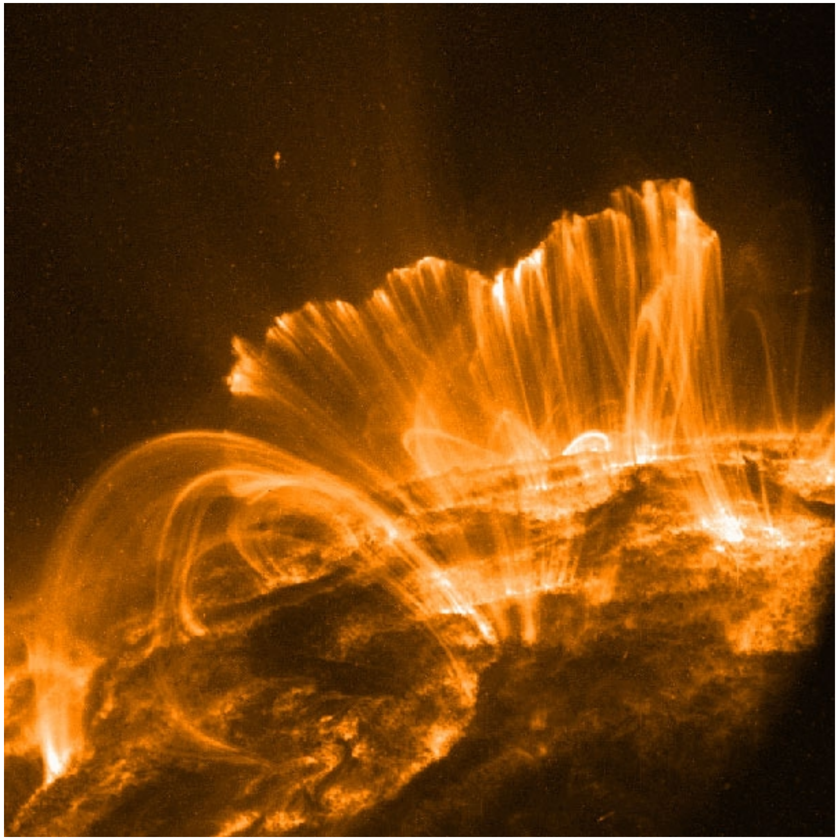
Solar corona: low plasma beta and magnetic resistivity

- Force-free magnetic fields
- Minimum energy state

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \;\Leftrightarrow\; \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

Beltrami field



NASA

Problem:
Find a force-free state for a magnetic field with given topology.

Here:
Numerical method for finding such states.

Ideal Field Relaxation

Ideal induction eq.: $$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

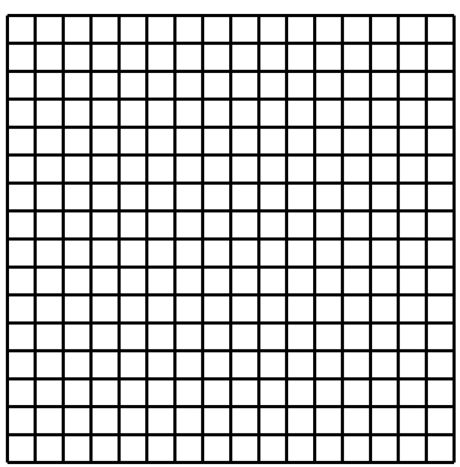
Frozen in magnetic field.

(Batchelor, 1950)

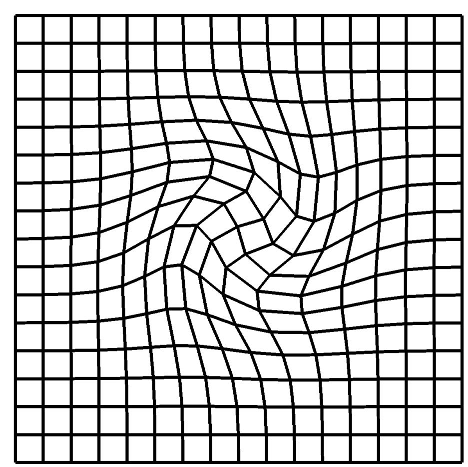
But: Numerical diffusion in finite difference Eulerian codes.

Solution: Lagrangian description of moving fluid particles:

$\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$



$\mathbf{x}(\mathbf{X}, t)$



Ideal Field Relaxation

Field evolution: $$B_i(\mathbf{X},t) = \frac{1}{\Delta}\sum_{j=1}^{3}\frac{\partial x_i}{\partial X_j}B_j(\mathbf{X},0)$$

$$\Delta = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

Preserves topology and divergence-freeness.

Grid evolution: $$\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \mathbf{u}\left(\mathbf{x}(\mathbf{X},t),t\right)$$

Magneto-frictional term: $\mathbf{u} = \gamma \mathbf{J}\times\mathbf{B}$ $\quad \mathbf{J} = \nabla\times\mathbf{B}$

$$\Rightarrow \frac{\mathrm{d}E_{\mathrm{M}}}{\mathrm{d}t} < 0$$

(Craig and Sneyd 1986)

Numerical Curl Operator

Compute $\mathbf{J} = \nabla \times \mathbf{B}$ on a distorted grid:

$$\frac{\partial B_i}{\partial x_j} = X_{\alpha,j}(x_{i,\alpha\beta} B^0_\beta \Delta^{-1} + x_{i,\beta} B^0_{\beta,\alpha} \Delta^{-1} - x_{i,\beta} B^0_\beta \Delta^{-2} \Delta_{,\alpha})$$

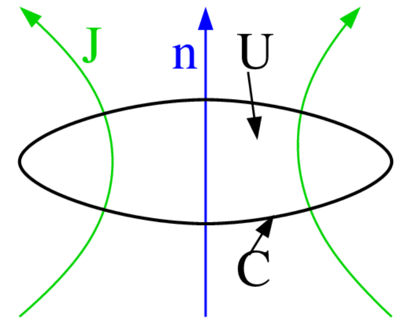
(Craig and Sneyd 1986)

$$B^0_i = B_i(0)$$

- Multiplication of several terms leads to high numerical errors.
- Current not divergence free: $\nabla \cdot \mathbf{J} \neq 0$
- Only reaching a certain force-freeness. *(Pontin et al. 2009)*

Mimetic Numerical Operators

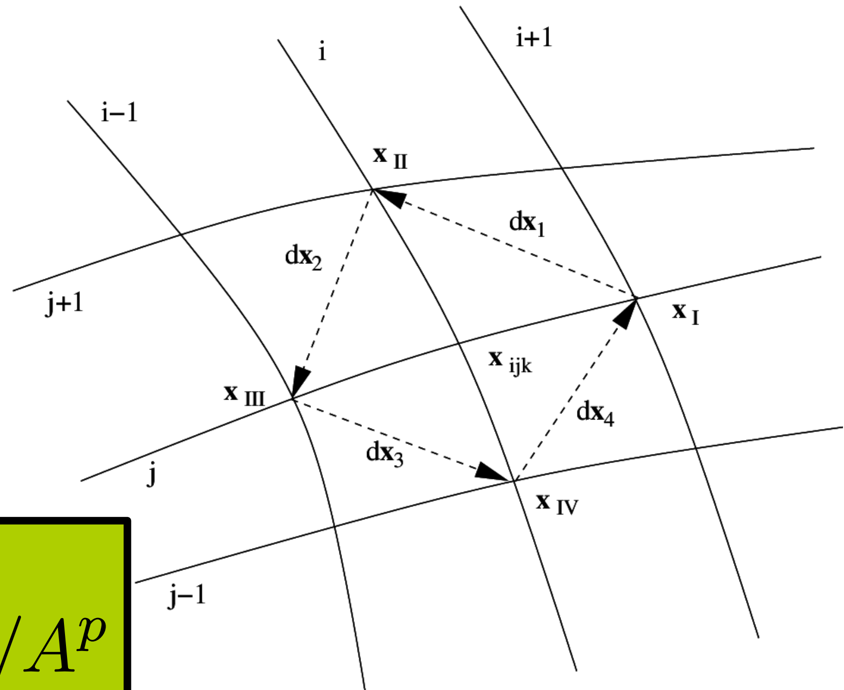
$$I = \int_U \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}S = \oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r}$$



Discretized:

$$I \approx \mathbf{J}(\mathbf{X}_{ijk})\cdot\mathbf{n}A = \sum_{r=1}^{4}\mathbf{B}_r\cdot\mathrm{d}\mathbf{x}_r$$

$$\mathbf{J}(\mathbf{X}_U) \approx \mathbf{J}(\mathbf{X}_{\mathbf{ijk}}), \quad \mathbf{X}_U \in U$$



3 planes will give 3 l.i. normal vectors:

$$\boxed{I^p = \mathbf{J}(\mathbf{X}_{ijk})\cdot\mathbf{n}^p = \sum_{r=1}^{4}\mathbf{B}_r^p\cdot\mathrm{d}\mathbf{x}_r/A^p}$$

Inversion yields $\mathbf{J}$ with $\nabla\cdot\mathbf{J} = 0$.

(Hyman, Shashkov 1997)

$$\nabla_{\mathrm{M}}\times\nabla_{\mathrm{M}}\phi = 0$$

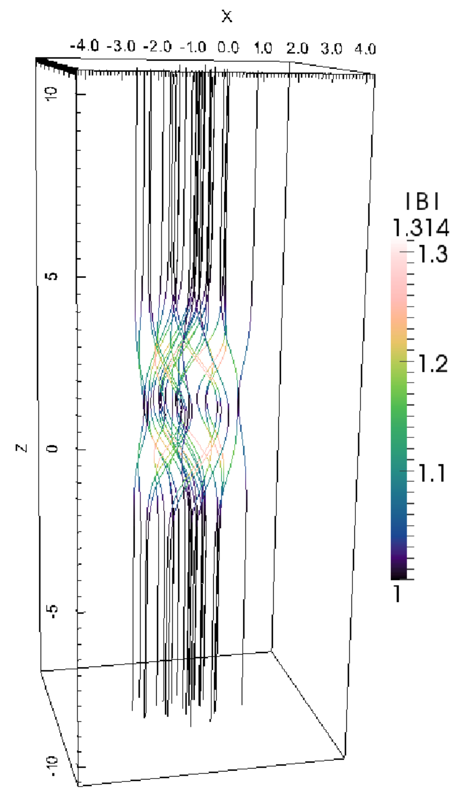
$$\nabla_{\mathrm{M}}\cdot\nabla_{\mathrm{M}}\times\mathbf{A} = 0$$

Simulations

- GPU code GLEMuR (**G**pu-based **L**agrangian mim**E**tic **M**agnetic **R**elaxation)
- line tied boundaries
- mimetic vs. classic

(Candelaresi et al. 2014)

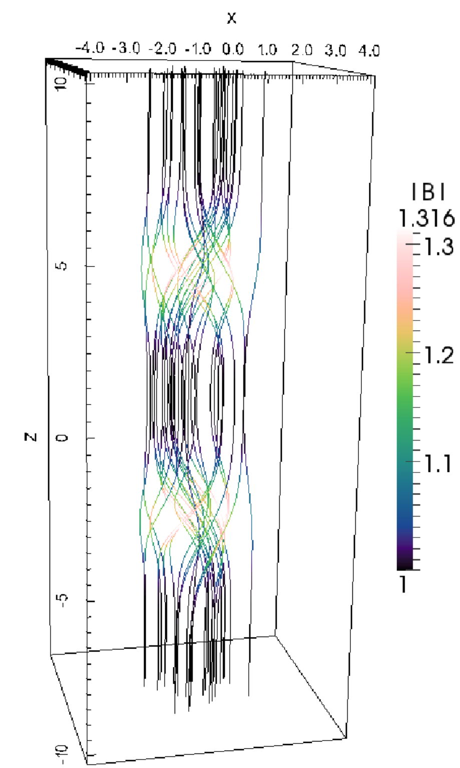
Nvidia Tesla K40



we know:

$$\lim_{t\to\infty} \mathbf{B}(t)$$

$$\lim_{t\to\infty} \mathbf{x}(t)$$



we know:

$$\lim_{t\to\infty} \mathbf{B}(t)$$

Quality Parameters

Deviation from the expected relaxed state:

$$\sigma_{\mathbf{x}} = \sqrt{\frac{1}{N}\sum_{ijk}(\mathbf{x}(\mathbf{X}_{ijk}) - \mathbf{x}_{\mathrm{relax}}(\mathbf{X}_{ijk}))^2}$$

$$\sigma_{\mathbf{B}} = \sqrt{\frac{1}{N}\sum_{ijk}(\mathbf{B}(\mathbf{X}_{ijk}) - \mathbf{B}_{\mathrm{relax}}(\mathbf{X}_{ijk}))^2}$$

Free magnetic energy:

$$E_{\mathrm{M}}^{\mathrm{free}} = E_{\mathrm{M}} - E_{\mathrm{M}}^{\mathrm{bkg}}$$

$$E_{\mathrm{M}} = \int_V \mathbf{B}^2/2 \; \mathrm{d}V \qquad \mathbf{B}^{\mathrm{bkg}} = B_0 \hat{e}_z$$

Quality Parameters

For a force-free field: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

⇨ Force-free parameter does not change along field lines.

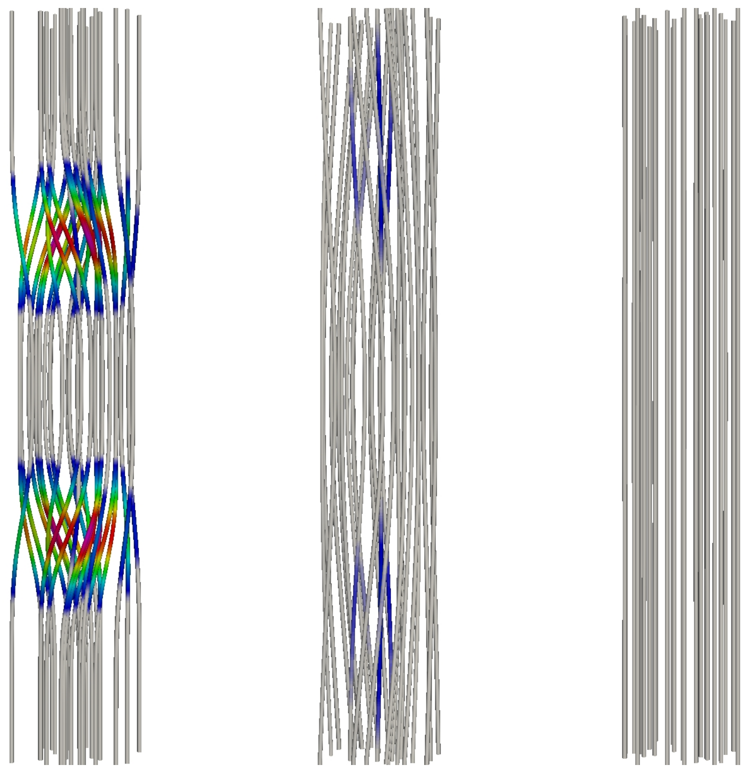
⇨ Measure the change of $\alpha^* = \dfrac{\mathbf{J} \cdot \mathbf{B}}{\mathbf{B}^2}$ along field lines:

$$\epsilon^* = \max_{i,j} \left(a_r \frac{\alpha^*(\mathbf{X}_i) - \alpha^*(\mathbf{X}_j)}{|\mathbf{X}_i - \mathbf{X}_j|} \right); \quad \mathbf{X}_i, \mathbf{X}_j \in s_\alpha$$

Particular field line: $s_\alpha = \{(0,0,Z) : Z \in [-L_z/2, L_z/2]\}$

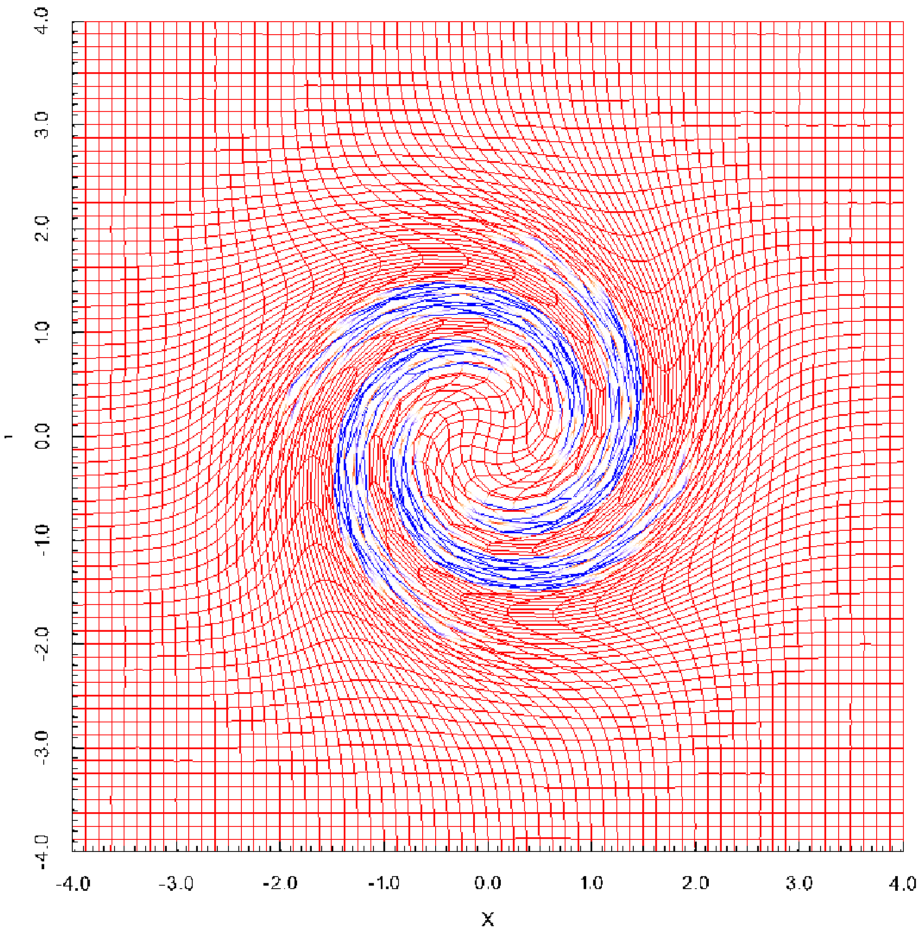
Field Relaxation

Magnetic streamlines:



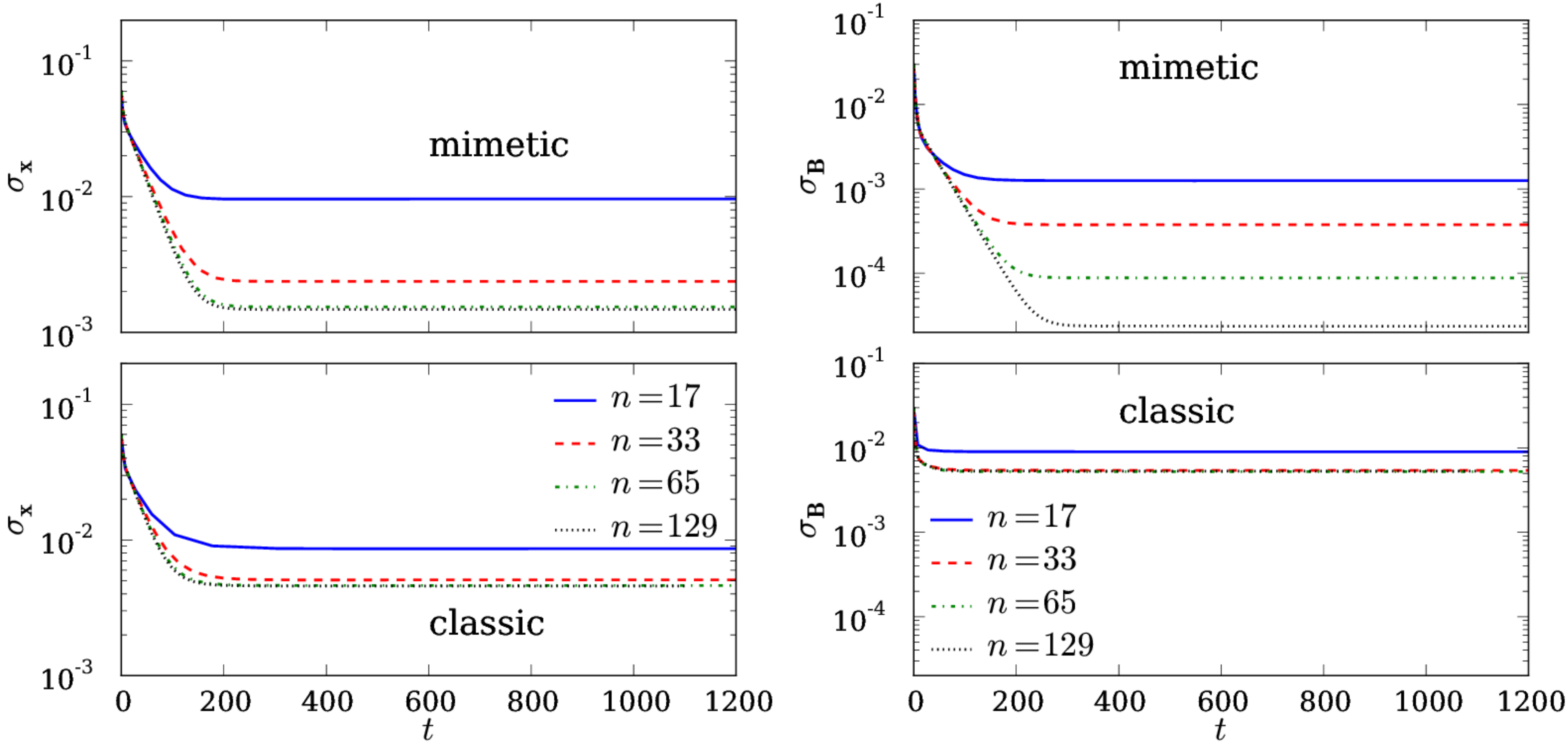
$t = 0$ $t = 10$ $t = 430$

Grid distortion at mid-plane:



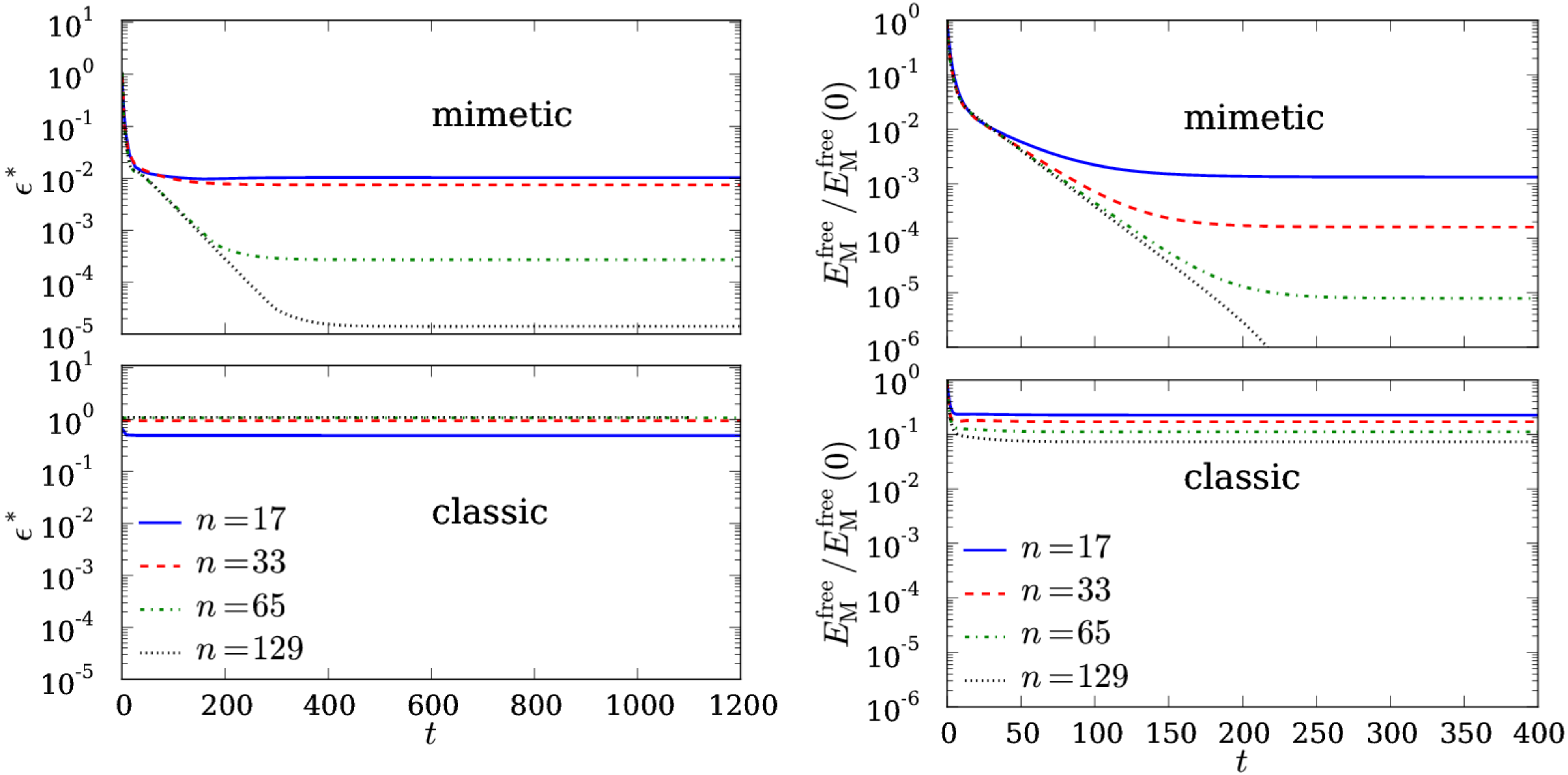
movie

Relaxation Quality



Closer to the analytical solution by 3 orders of magnitude.

Relaxation Quality



Closer to force-free state by 5 orders of magnitude.

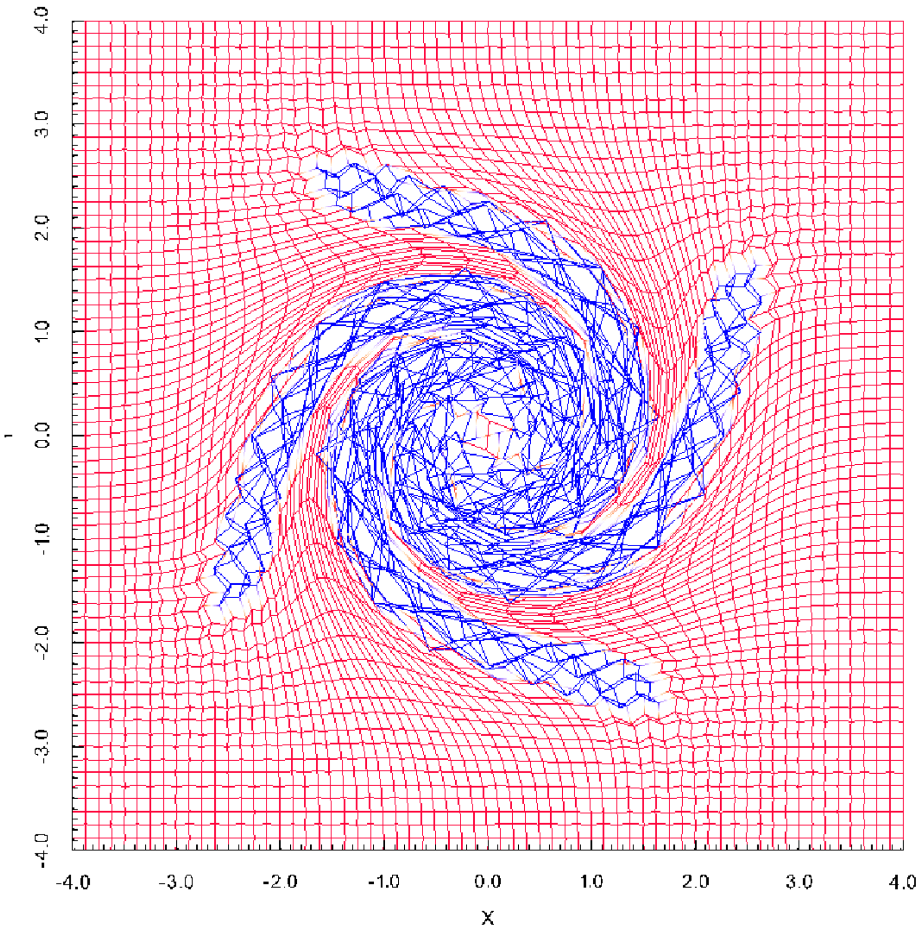
Performance Gain

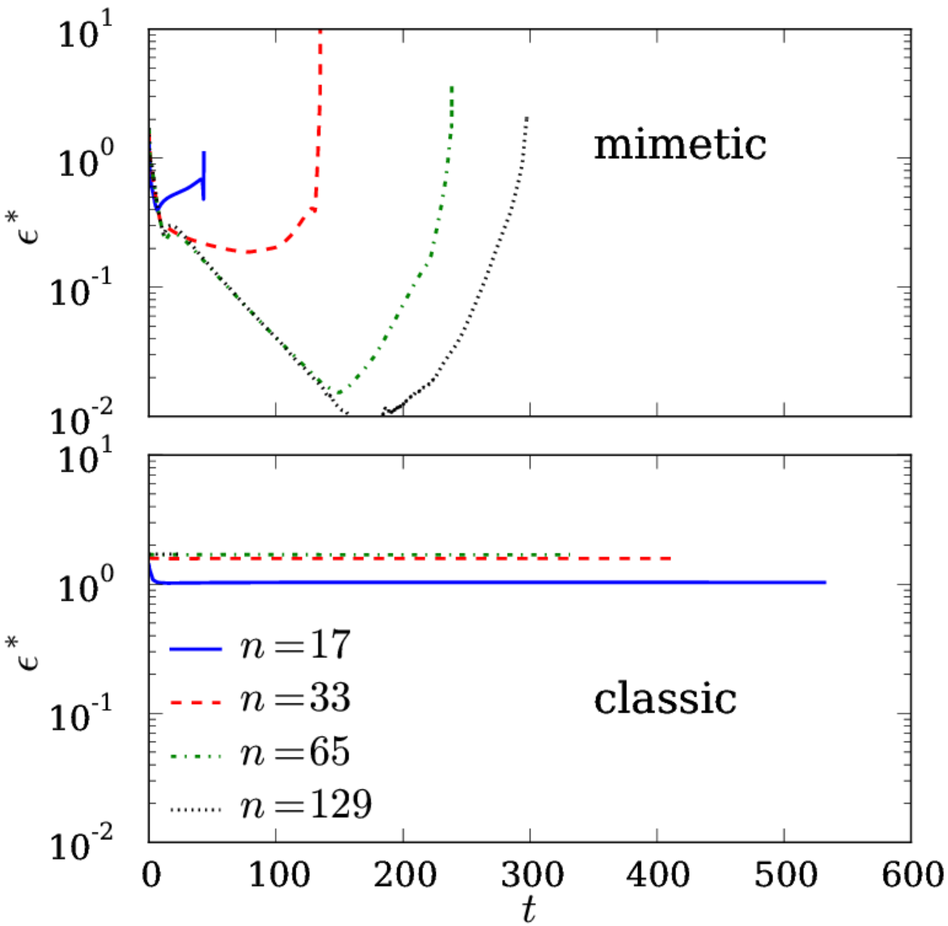
	mimetic vs. classic
floating point operations	1/2
computation time (gross)	1/2
previous code*	x100

*serial code using classical finite differences and an implicit solver

(Craig and Sneyd 1986)

Limitations





red: convex
blue: concave

For concave cells the method becomes unstable.
But: results before crash better than classic method.

Code Details

- written in C++
- running in GPUs
- VTK data format
- 6^{th} order Runge-Kutta time stepping
- periodic and line-tied boundaries
- post processing routines in Python

```
// compute the norm of JxB/B**2
__global__ void JxB_B2(REAL *B, REAL *J, REAL *JxB_B2, int dimX, int dimY, int dimZ) {
	int i = threadIdx.x + blockDim.x * blockIdx.x;
	int j = threadIdx.y + blockDim.y * blockIdx.y;
	int k = threadIdx.z + blockDim.z * blockIdx.z;
	int p = threadIdx.x;
	int q = threadIdx.y;
	int r = threadIdx.z;
	int l;
	REAL B2;

	// shared memory for faster communication, the size is assigned dynamically
	extern __shared__ REAL s[];
	REAL *Bs = s;                              // magnetic field
	REAL *Js = &s[3 * dimX * dimY * dimZ];      // electric current density
	REAL *JxBs = &Js[3 * dimX * dimY * dimZ];   // JxB

	// copy from global memory into shared memory
	if ((i < dev_p.nx) && (j < dev_p.ny) && (k < dev_p.nz)) {
		for (l = 0; l < 3; l++) {
			Bs[l + p*3 + q*dimX*3 + r*dimX*dimY*3] = B[l + (i+1)*3 + (j+1)*(dev_p.nx+2)*3 + (k+1)*(dev_p.nx+2)*(dev_p.ny+2)*3];
			Js[l + p*3 + q*dimX*3 + r*dimX*dimY*3] = J[l + i*3 + j*dev_p.nx*3 + k*dev_p.nx*dev_p.ny*3];
		}

		cross(&Js[0 + p*3 + q*dimX*3 + r*dimX*dimY*3],
		      &Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3],
		      &JxBs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3]);

		B2 = dot(&Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3], &Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3]);

		// return result into global memory
		JxB_B2[i + j*dev_p.nx + k*dev_p.nx*dev_p.ny] = norm(&JxBs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3])/B2;
	}
}
```

GLEMuR vs. PencilCode

	GLEMuR	PencilCode
data format	VTK	PC
language	C++	Fortran
change # cores	✓	✘
compile once	✓	✘
post processing	Python	IDL/Python
bash tools	✓	✓
GPU	✓	✘
CPU	✘	✓
general MHD	✘	✓

Similarities with the PencilCode

Fortran name lists

```
&comp
    nx = 33;  ny = 33;  nz = 33
/

&start
    Lx = 0.6;     Ly = 0.6;     Lz = 1.0
    Ox = -0.3;    Oy = -0.3;    Oz = -0.5
    bInit = "sheared"
    ampl = 1.
    initDist = "initShearX"
    initShear0 = 0.7
    initShearK = 1.
    fRestart = t
/
```

Bash commands

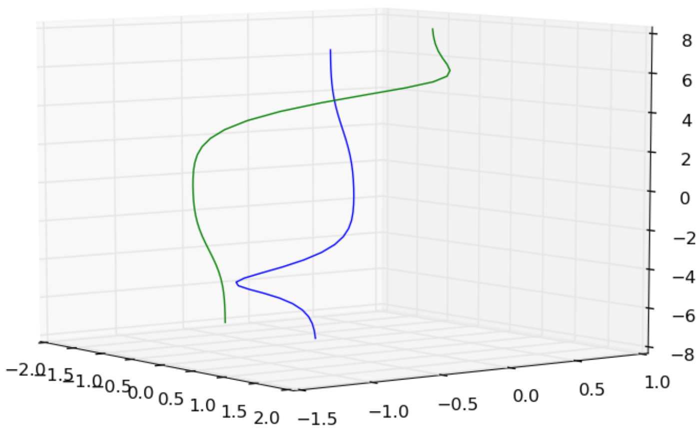
```
gm_ci_run
gm_inspectrun
gm_newrun
```

time_series.dat

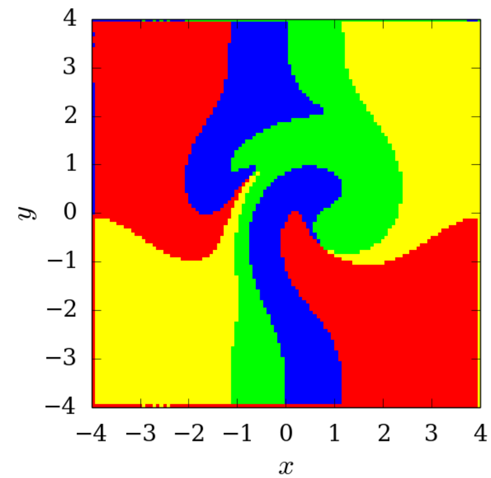
```
#     it           t          dt    maxDelta    JxB_B2Max  epsilonStar          B2       convex
       0 1.29540e-06 1.29540e-06 1.78814e-07 1.50932e+02 2.81160e+02 7.12394e-01 -1.00000e+00
       1 3.08508e-06 1.78967e-06 1.19209e-07 1.13175e+02 2.96738e+02 7.12170e-01 -1.00000e+00
       2 5.76647e-06 2.68139e-06 1.78814e-07 8.83429e+01 3.15884e+02 7.11882e-01 -1.00000e+00
       3 9.47096e-06 3.70449e-06 1.19209e-07 7.67879e+01 3.36120e+02 7.11536e-01 -1.00000e+00
       4 1.50212e-05 5.55028e-06 1.78814e-07 6.44194e+01 3.57402e+02 7.11085e-01 -1.00000e+00
       5 2.13638e-05 6.34253e-06 7.74860e-07 5.44002e+01 3.73753e+02 7.10636e-01 -1.00000e+00
```

Post-Processing

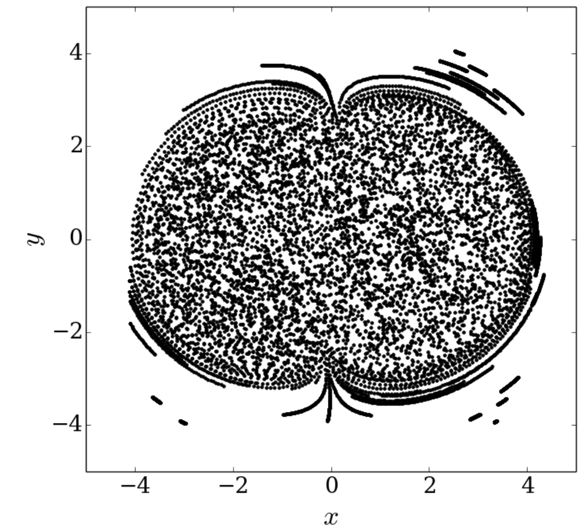
streamlines



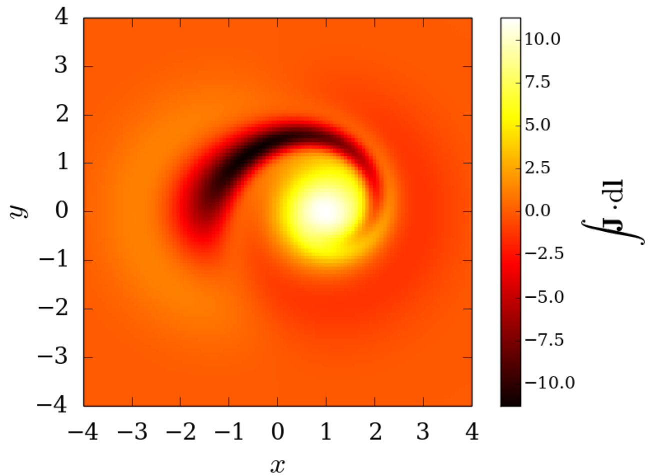
field line mapping



Poincaré maps



line integration



save and read as vtk file

```
s0 = gm.streamInit(tol = 0.01)
```

```
stream.vtk
```

```
sr = gm.readStream()
```

Outlook

- GLEMuR to PC data conversion
- More physics (multi purpose)
- Run on GPU clusters
- PencilCode on GPUs?

Conclusions

- Lagrangian numerical scheme for ideal evolution.
- Preserving field line topology.
- Mimetic methods more capable of producing force-free fields.
- GLEMuR code running on GPUs.
- Performance gain of x2 compared to classical approach.
- GLEMuR vs. PencilCode
- Design features from the PencilCode

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