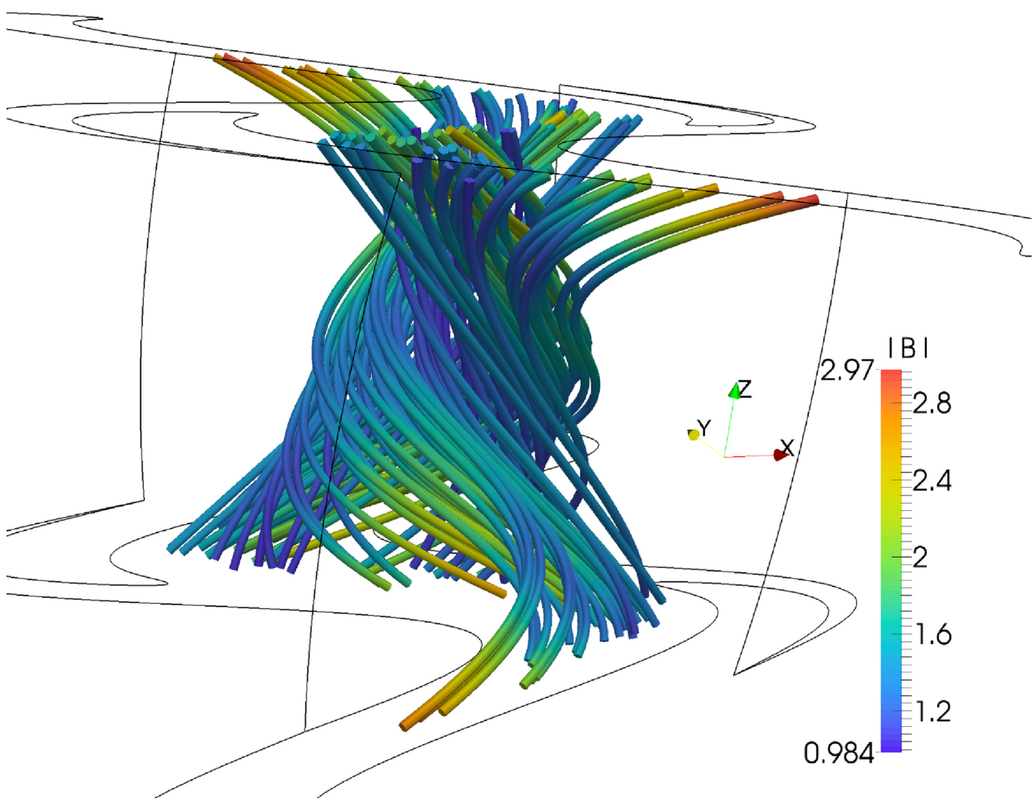
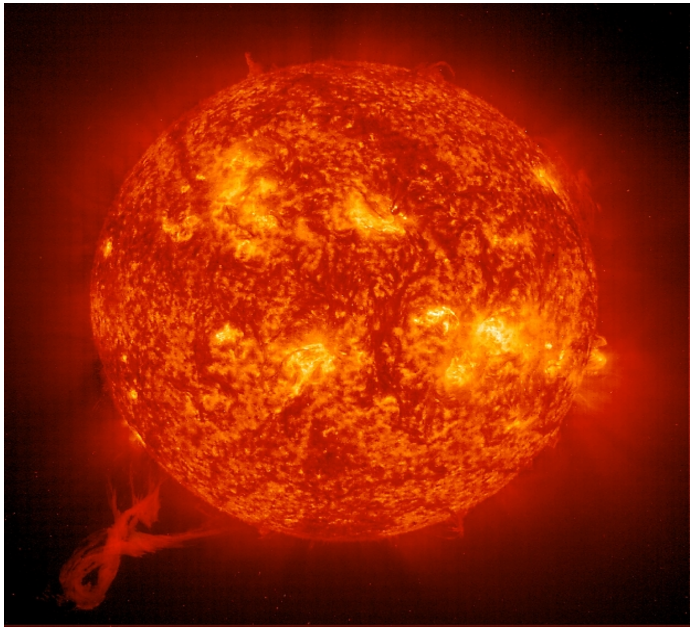
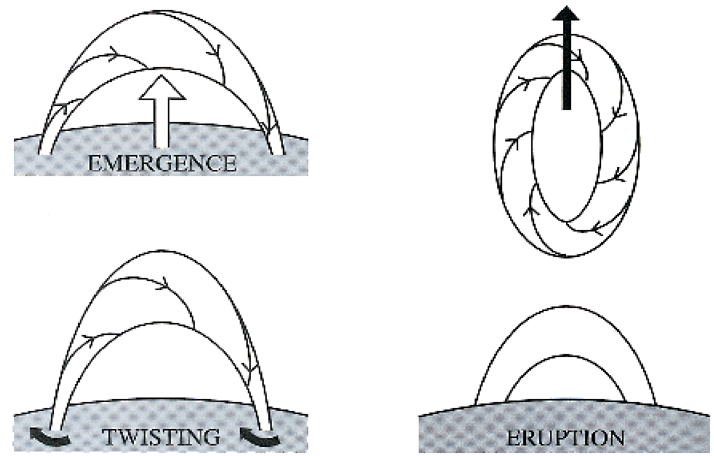
Topology in Magnetohydrodynamics

Simon Candelaresi, David MacTaggart

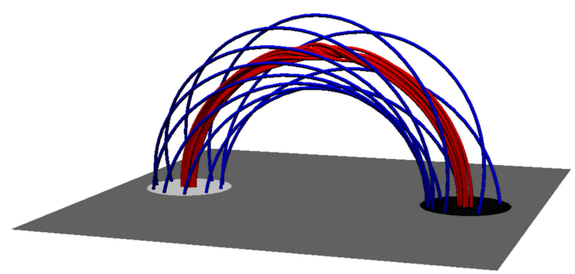


Twisted Magnetic Fields

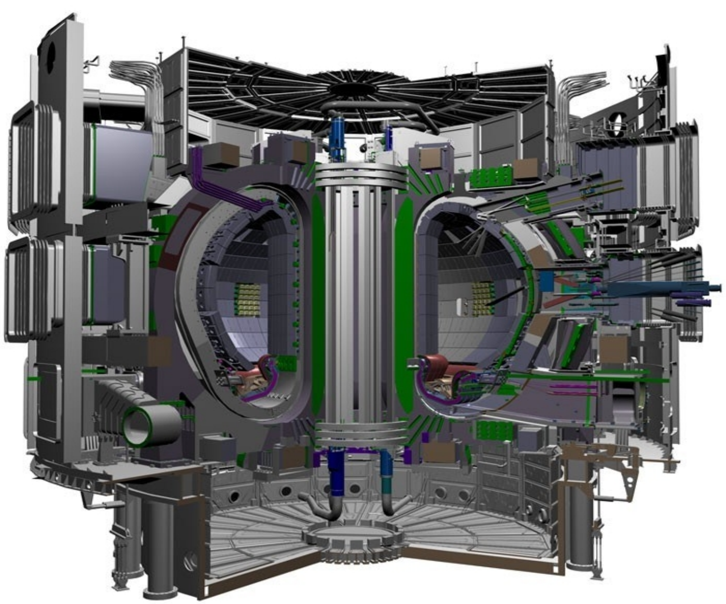




Twisted fields are more likely to erupt *(Canfield et al. 1999)*.

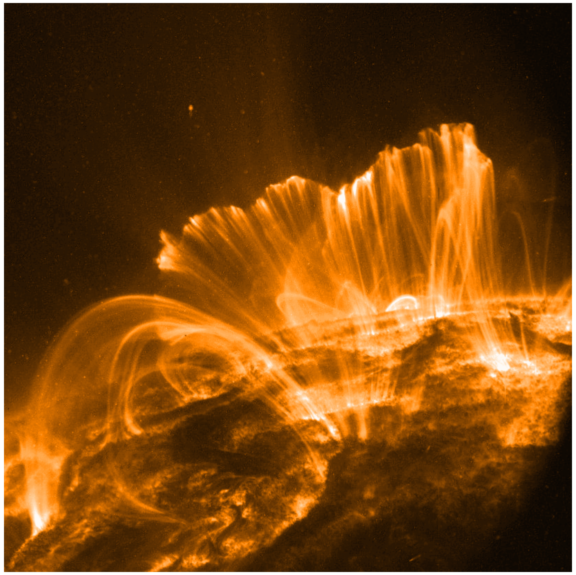


(MacTaggart, 2020)

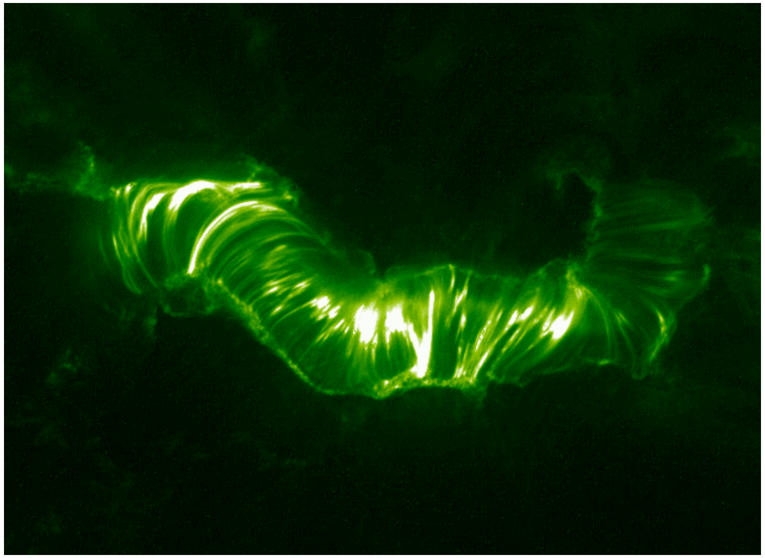


Twist increases the stability of magnetic fields in tokamaks.

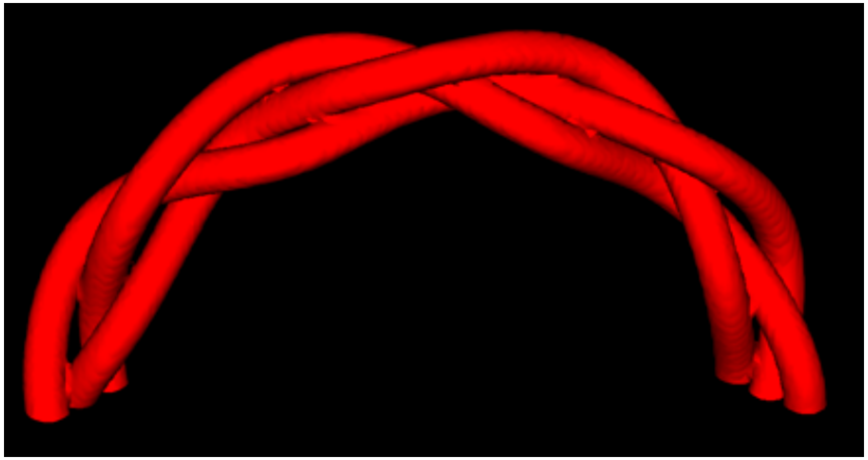
Solar Magnetic Field



(Trace)



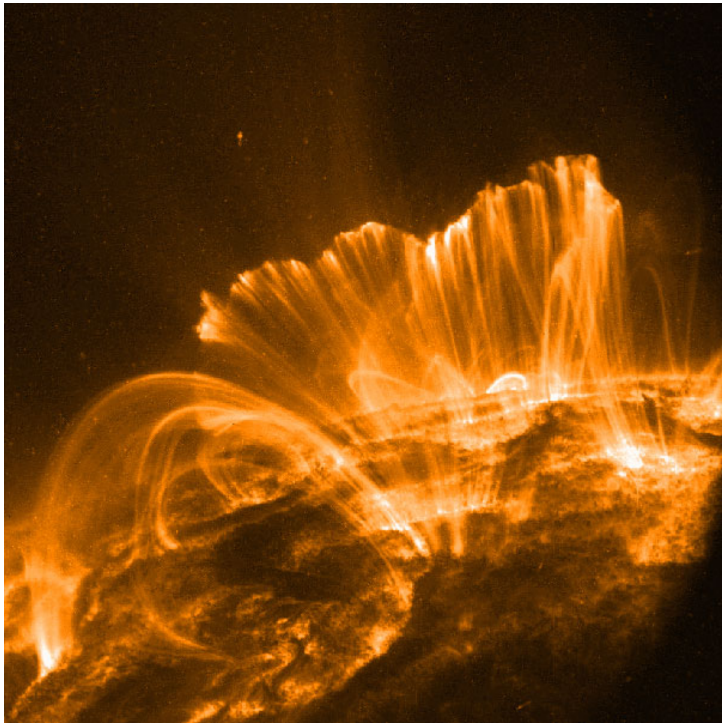
(Trace)

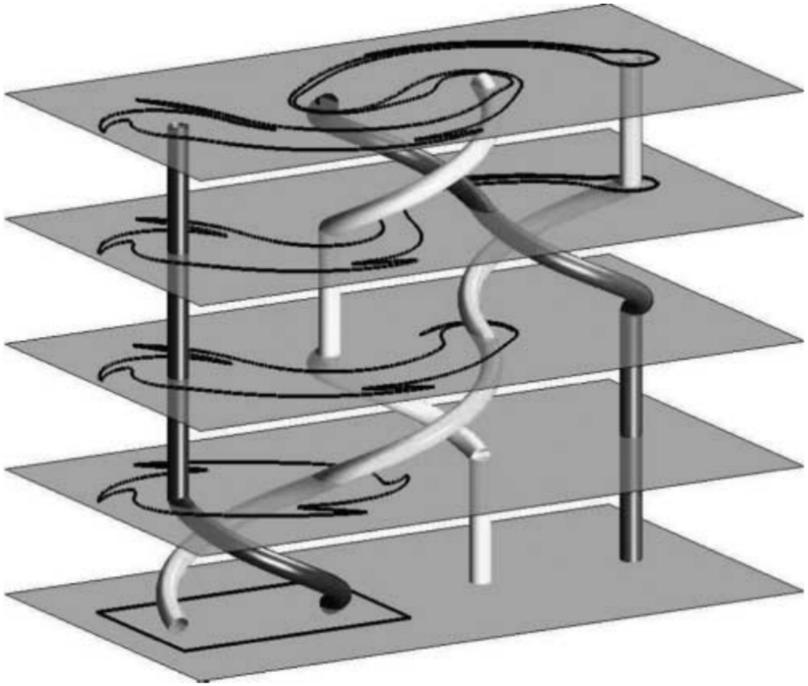


- Twisted flux tubes may rise to the corona. *(Prior and MacTaggart 2016)*.

Coronal Magnetic Fields

NASA





(Thiffeault et al. 2006)

➡ Field line tangling in solar magnetic fields.

Magnetohydrodynamics

Non-relativistic + isothermal + compressible + viscous medium.

$$\frac{\mathrm{D}\ln\rho}{\mathrm{D}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{u}$$ conservation of mass

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -c_{\mathrm{S}}^2\boldsymbol{\nabla}\ln\rho + \boldsymbol{J}\times\boldsymbol{B}/\rho + \boldsymbol{F}_{\mathrm{visc}}$$ momentum eq.

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) + \eta\nabla^2\boldsymbol{B}$$ induction eq.

$$\left(\frac{\partial\boldsymbol{A}}{\partial t} = \boldsymbol{u}\times\boldsymbol{B} + \eta\nabla^2\boldsymbol{A}\right)$$

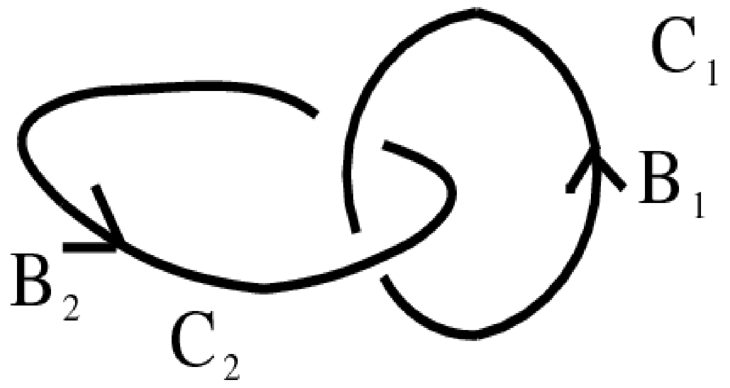
$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} \qquad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \qquad \boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{B}$$

Magnetic Helicity

Gauss linking number:

$$\mathrm{lk}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \underline{\frac{\partial \boldsymbol{r}_1(t_1)}{\partial t_1}} \times \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} \cdot \underline{\frac{\partial \boldsymbol{r}_2(t_2)}{\partial t_2}} \; \mathrm{d}t_1 \; \mathrm{d}t_2$$

Magnetic helicity:



$$H_\mathrm{M} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \; \mathrm{d}V$$

Biot-Savart:

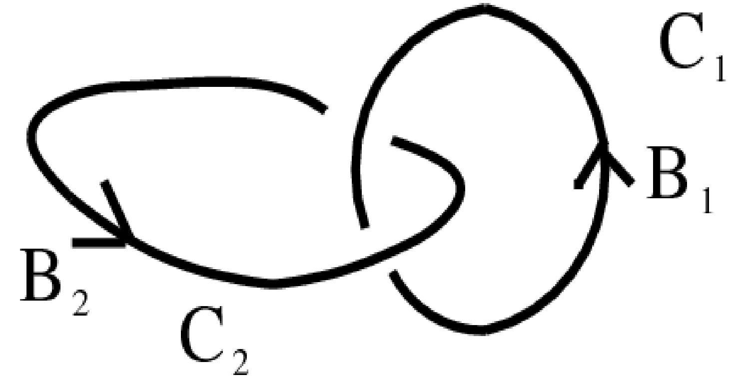
$$\boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi} \int_V \boldsymbol{B}(\boldsymbol{x}') \times \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \; \mathrm{d}^3\boldsymbol{x}'$$

$$H_\mathrm{m} = \frac{1}{4\pi} \int_V \int_V \underline{\boldsymbol{B}(\boldsymbol{x}')} \times \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \cdot \underline{\boldsymbol{B}(\boldsymbol{x})} \; \mathrm{d}^3\boldsymbol{x}' \; \mathrm{d}^3\boldsymbol{x}$$

Magnetic Helicity

$$H_{\mathrm{M}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 2n\phi_1\phi_2$$

$$\phi_i = \int_{S_i} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$



$n =$ number of mutual linking

Conservation of magnetic helicity:

$$\lim_{\eta \to 0} \frac{\partial}{\partial t} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = 0$$

$\eta =$ magnetic resistivity

Realizability condition:

$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

Magnetic energy is bound from below by magnetic helicity.

Ideal MHD

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

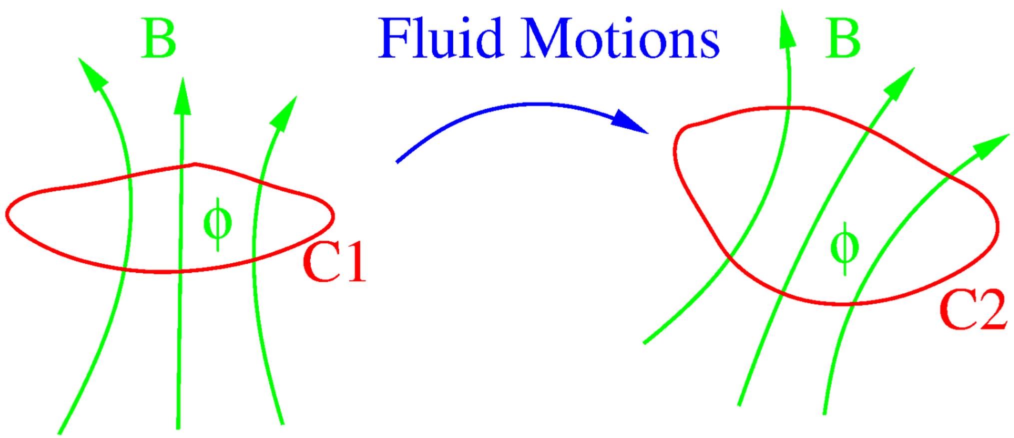
1-form: $\alpha = A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z$

2-form: $\beta = B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y$

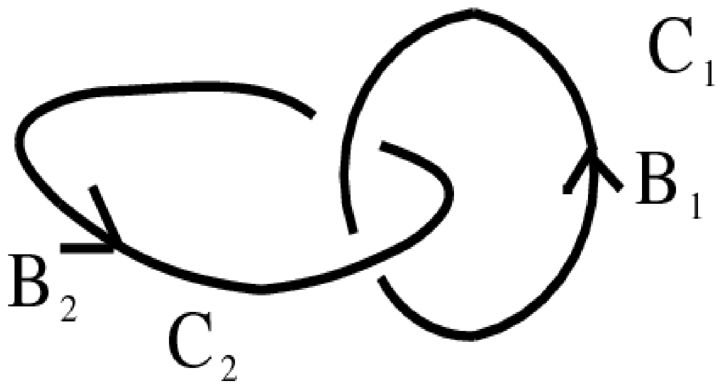
3-form: $h = (A_x B_x + A_y B_y + A_z B_z) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

Induction equation -> Lie-transport: $\frac{\partial}{\partial t}\beta(\boldsymbol{r}, t) + \mathcal{L}_{\boldsymbol{u}}\beta(\boldsymbol{r}, t) = 0$

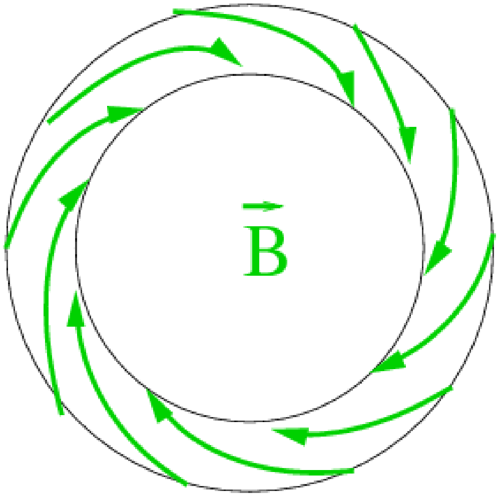
Magnetic field is *frozen-in* to the fluid.



Topologies of Magnetic Fields

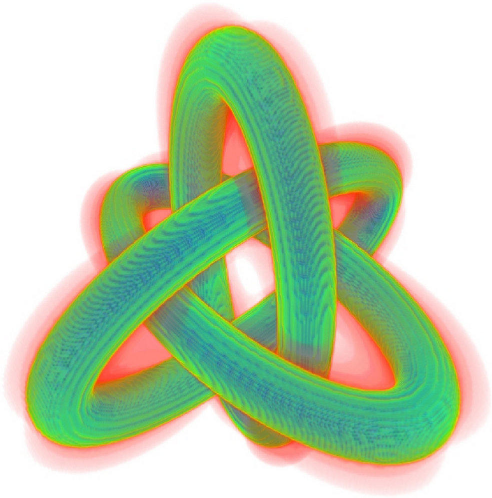


Hopf link

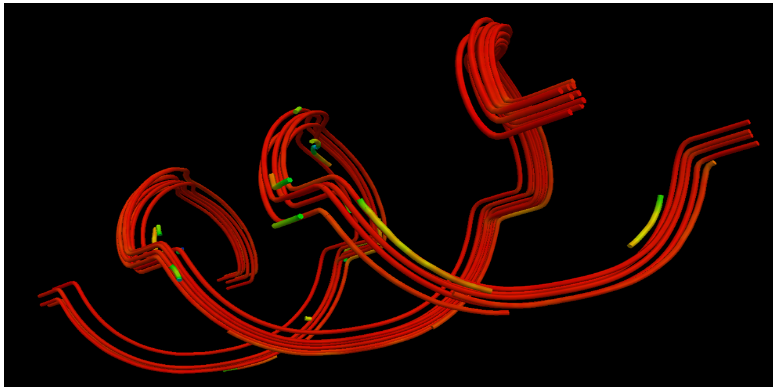


twisted field

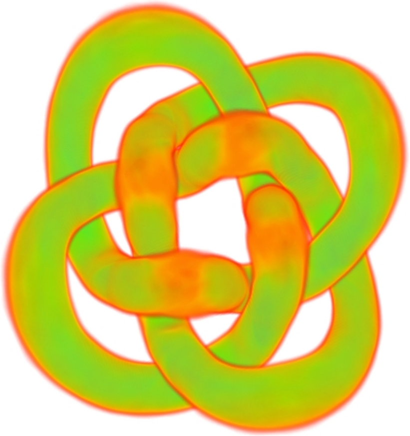
trefoil knot



Borromean rings



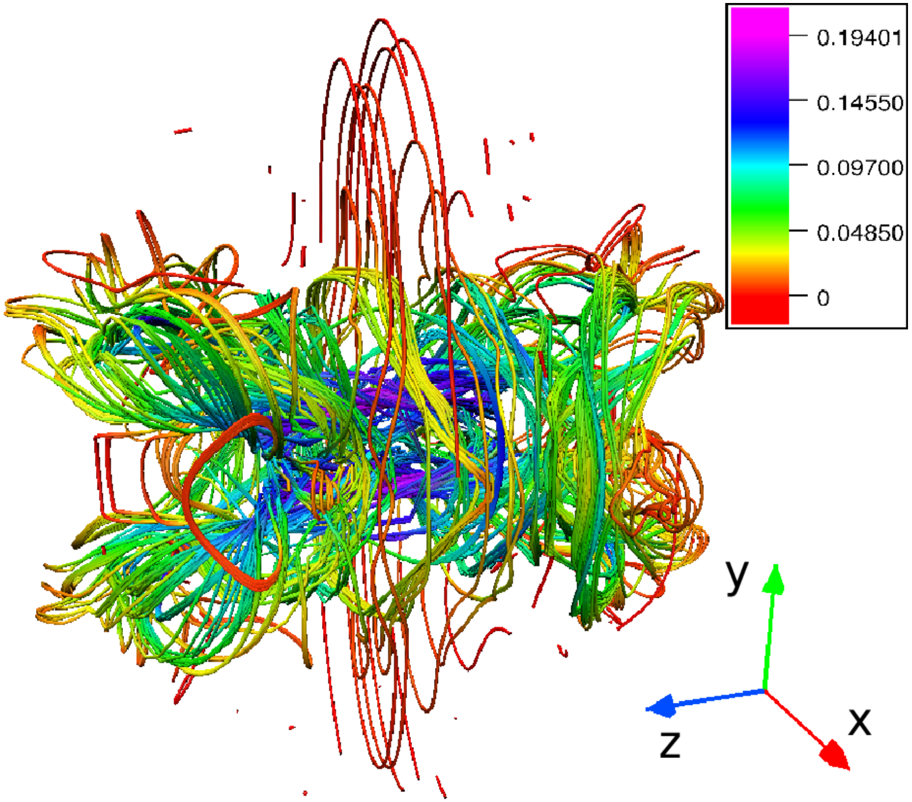
magnetic braid



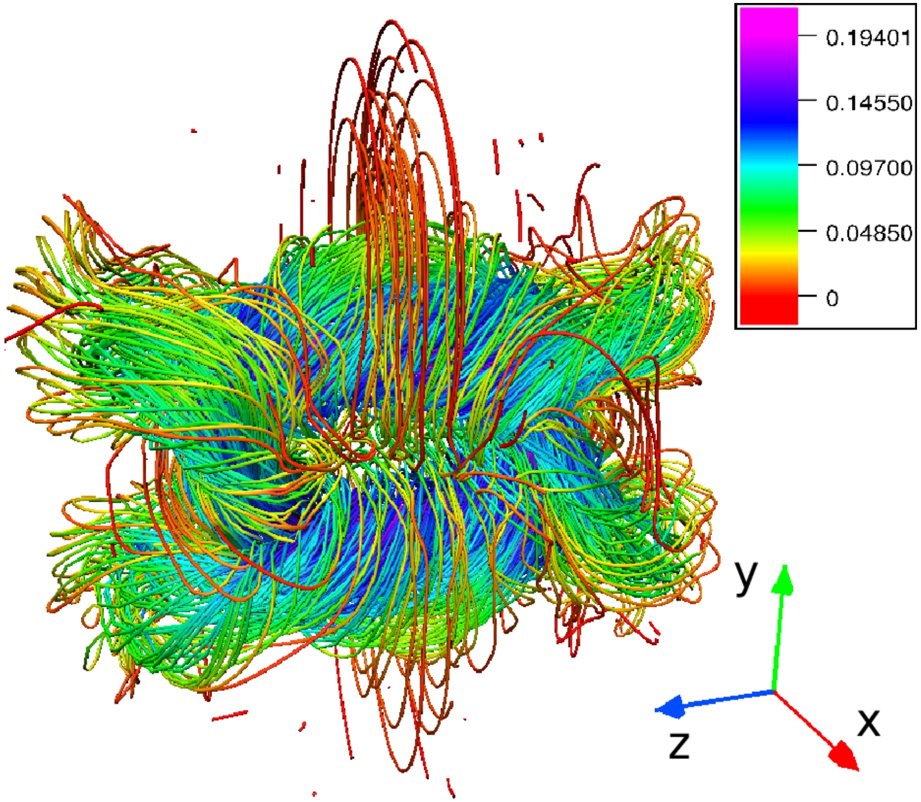
IUCAA knot

Interlocked Flux Rings

$\tau = 4$

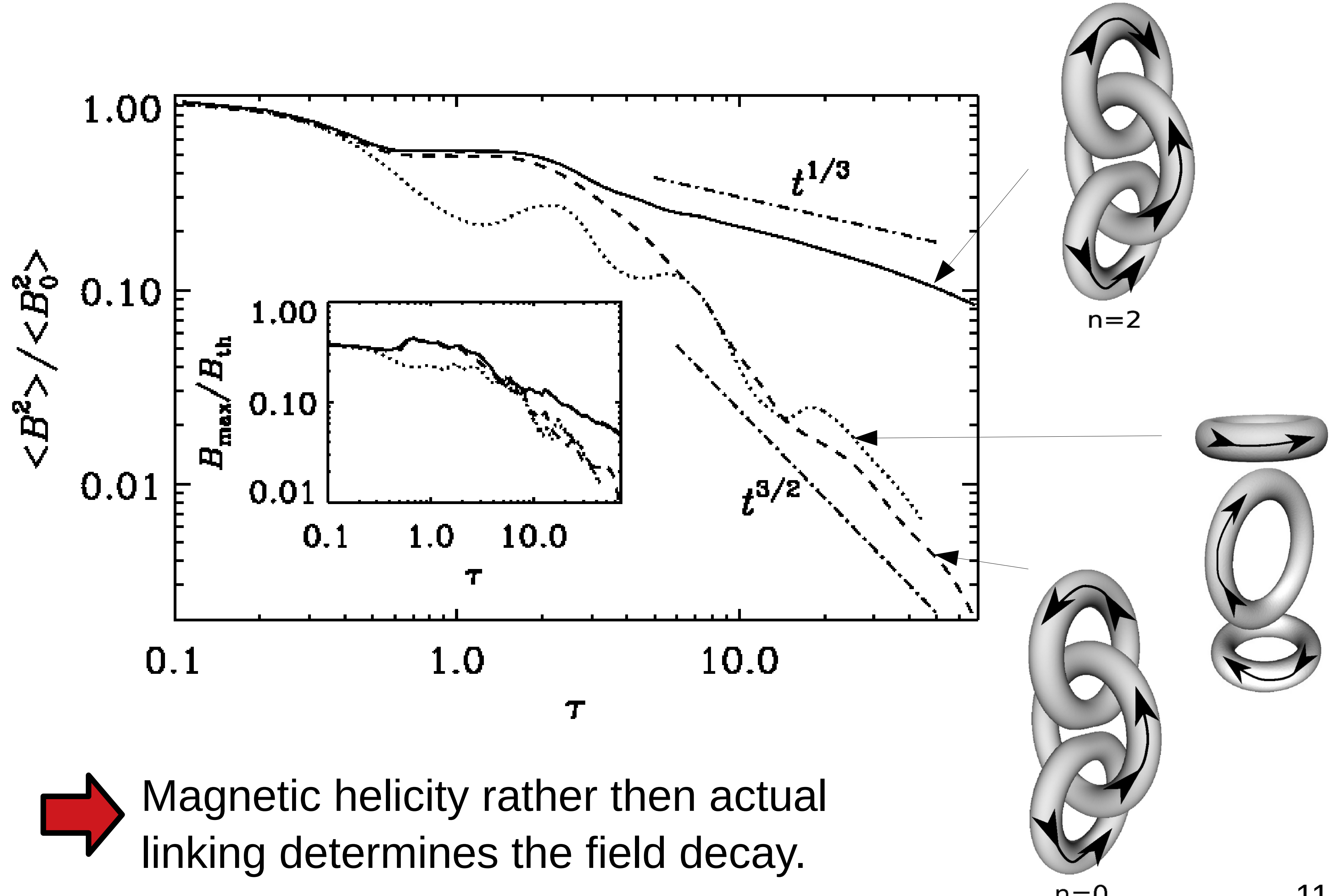


$H_{\mathrm{M}} = 0$



$H_{\mathrm{M}} \neq 0$

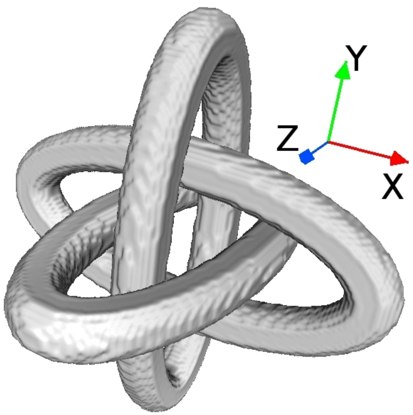
Interlocked Flux Rings



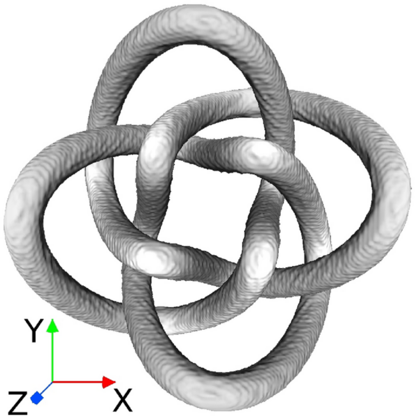
➡ Magnetic helicity rather then actual linking determines the field decay.

IUCAA Knot and Borromean Rings

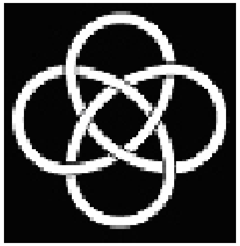
- Is magnetic helicity sufficient?
- Higher order invariants?



Borromean rings



IUCAA knot

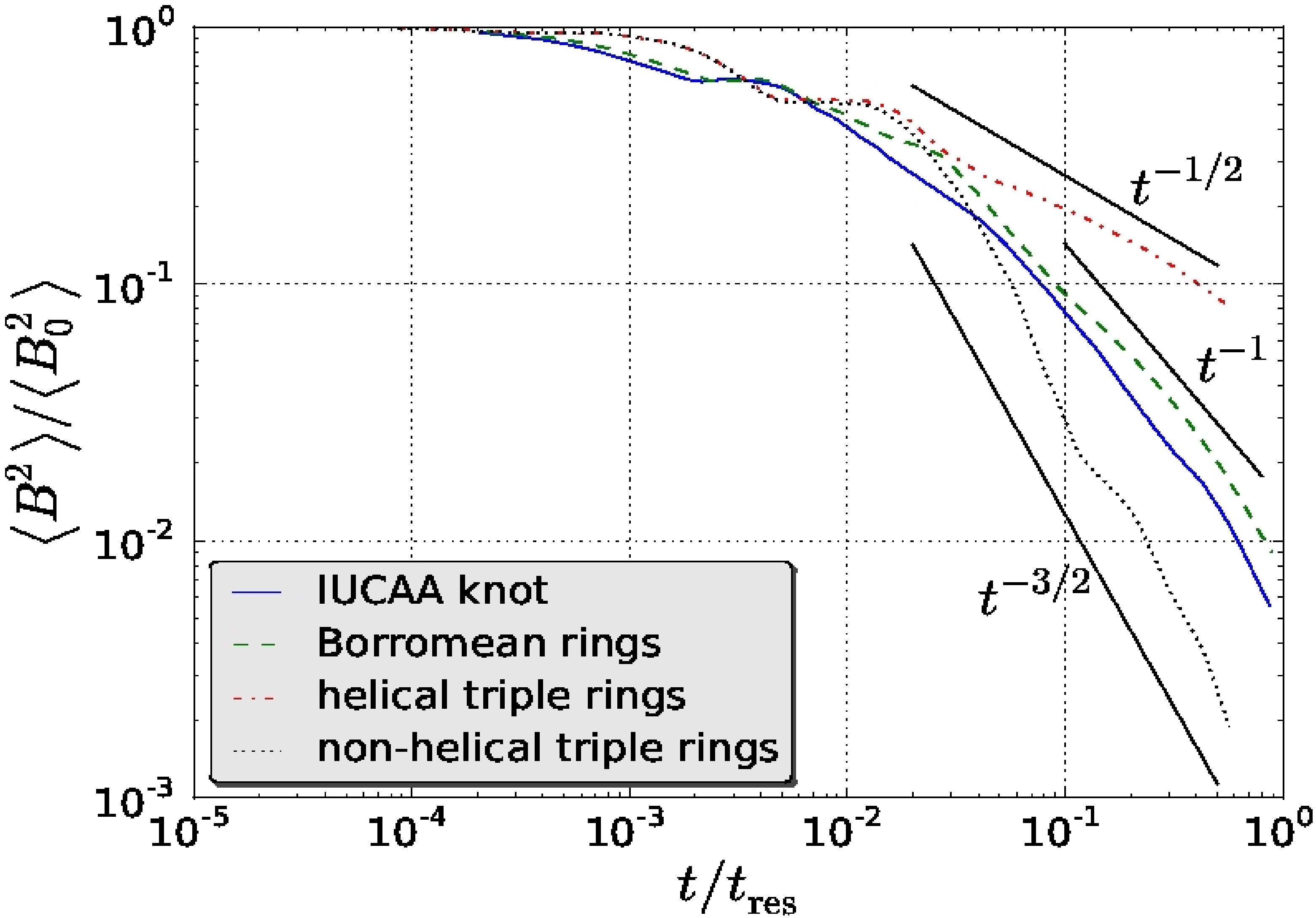


$$H_{\rm M} = 0$$

(Candelaresi and Brandenburg 2011)

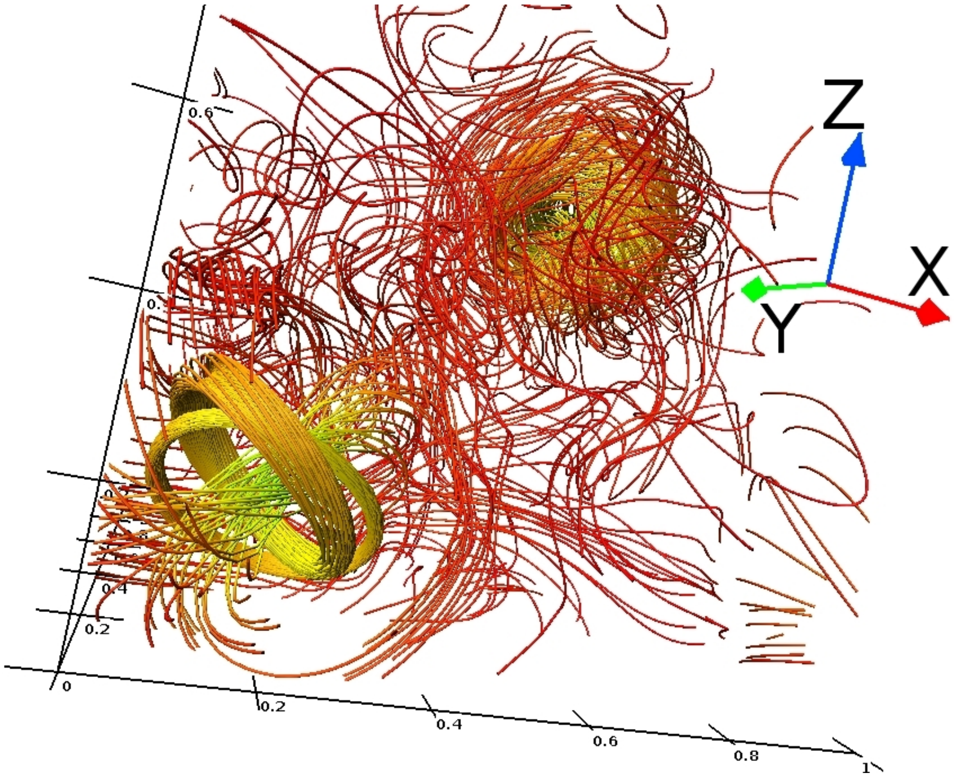
IUCAA = The Inter-University Centre for Astronomy and Astrophysics, Pune, India

Magnetic Energy Decay

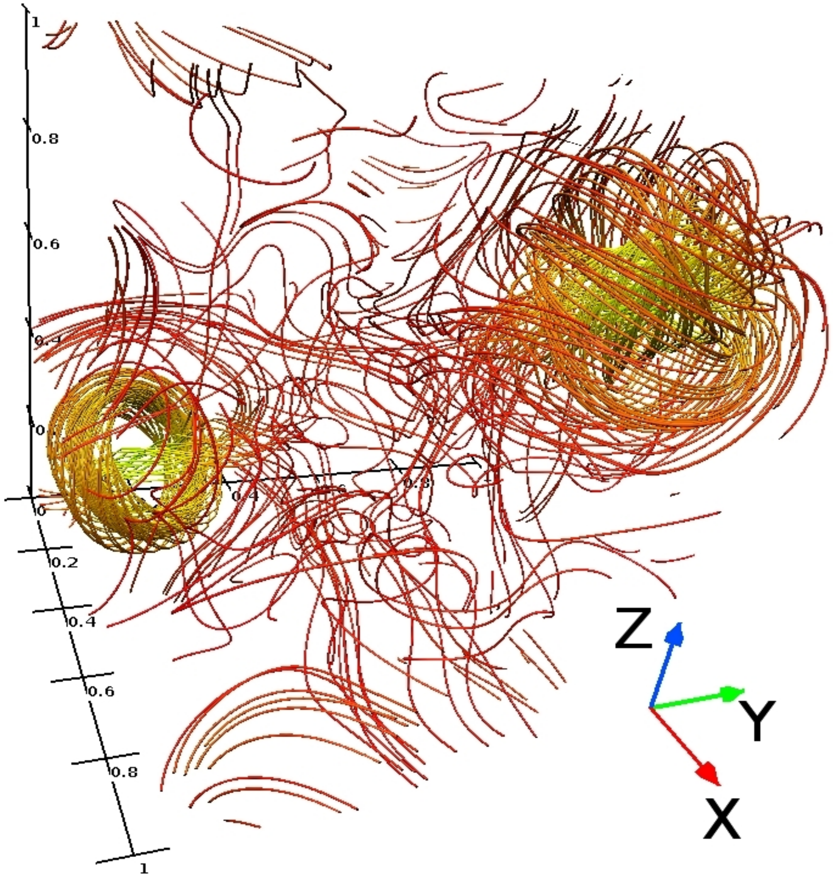


Higher order invariants?

Reconnection characteristics



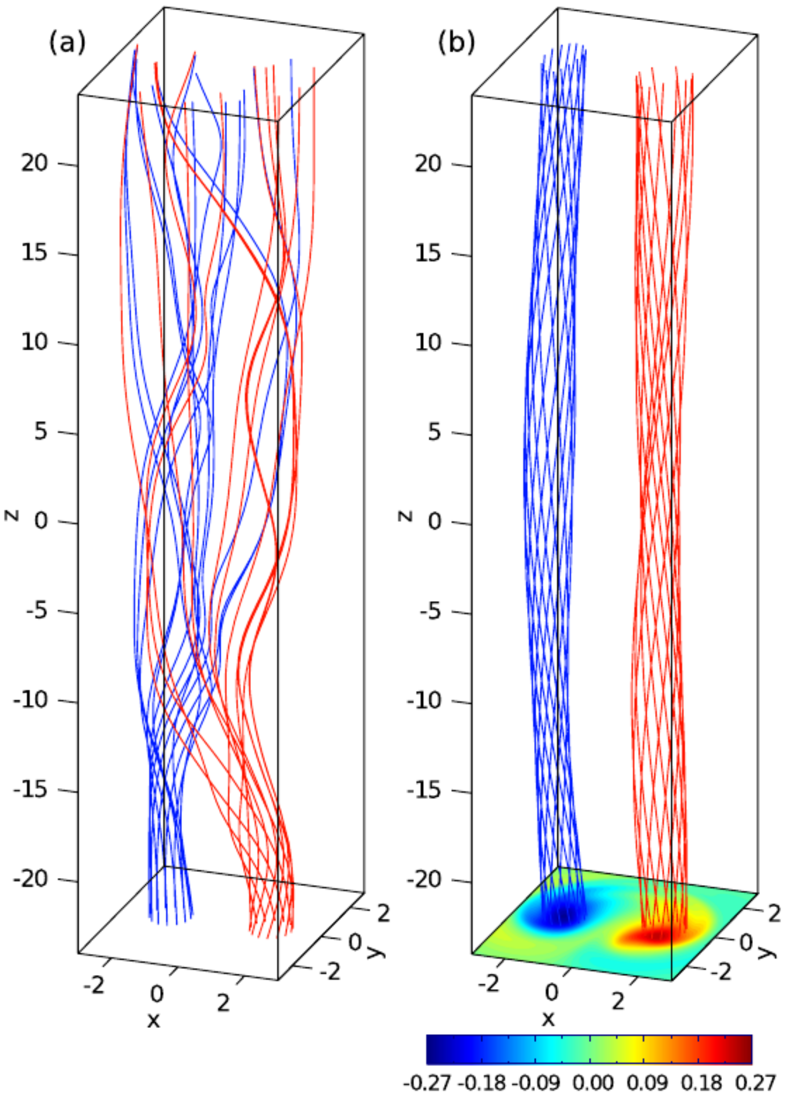
$t = 70$



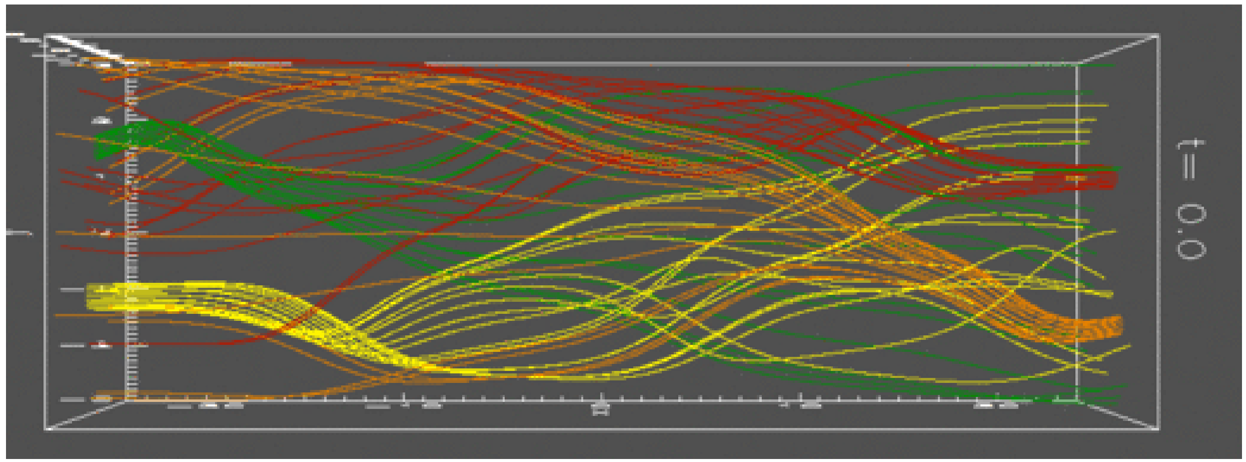
$t = 78$

3 rings ➡ Twisted ring + interlocked rings ➡ 2 twisted rings

Magnetic Braid



(Yeates 2011)



(Wilmot-Smith 2010)

- Periodic braid topologically equivalent to Borromean rings.
- Separation into two twisted field regions.
- Conserved invariants like fixed point index and field line helicity.

Stability criteria

	constraint	equilibrium
Woltjer (1958):	$\frac{\partial}{\partial t}\int_V \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V = 0$	$\boldsymbol{\nabla}\times\mathbf{B} = \alpha\mathbf{B}$
Taylor (1974):	$\frac{\partial}{\partial t}\int_{\tilde{V}} \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V = 0$	$\boldsymbol{\nabla}\times\mathbf{B} = \alpha(a,b)\mathbf{B}$

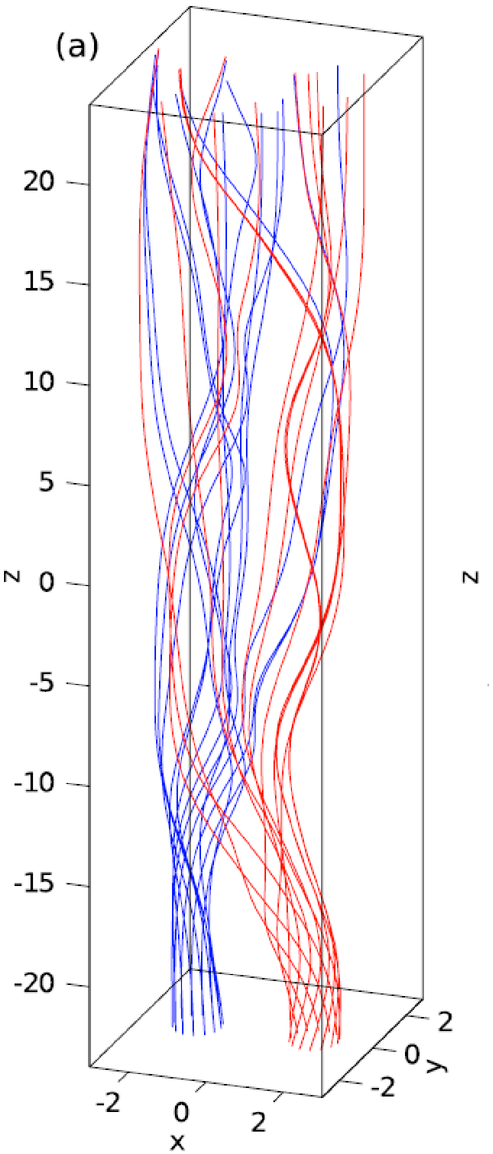
constant along field line

V = total volume $\tilde{V}$ = volume along magnetic field line

Taylor state not reached due to fixed point conservation.

(Yeates et al. 2011)

Fixed Point Index



(Yeates et al. 2011)

Trace magnetic field lines from z_0 to z.

mapping: $(x, y) \rightarrow \mathbf{F}_z(x, y)$

fixed points: $\mathbf{F}_1(x, y) = (x, y)$

Color coding:

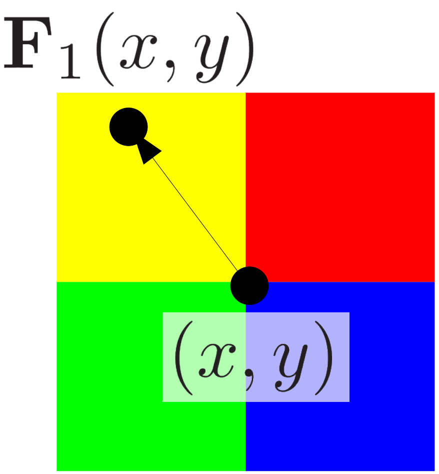
Compare (x, y) with $\mathbf{F}_1(x, y)$:

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y > y$ ⇨ red

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y > y$ ⇨ yellow

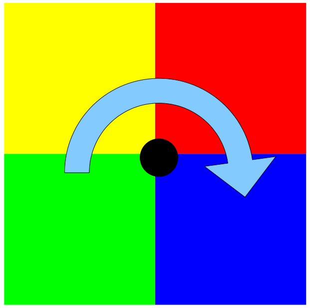
$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y < y$ ⇨ green

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y < y$ ⇨ blue

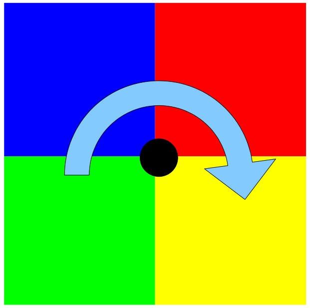


Fixed Point Index

Sign t_i of fixed point i :

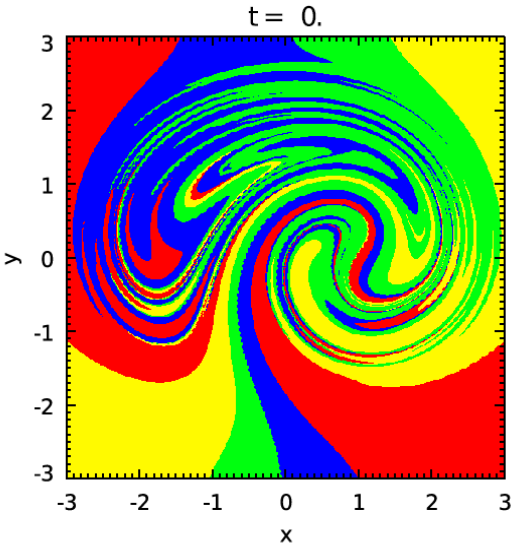


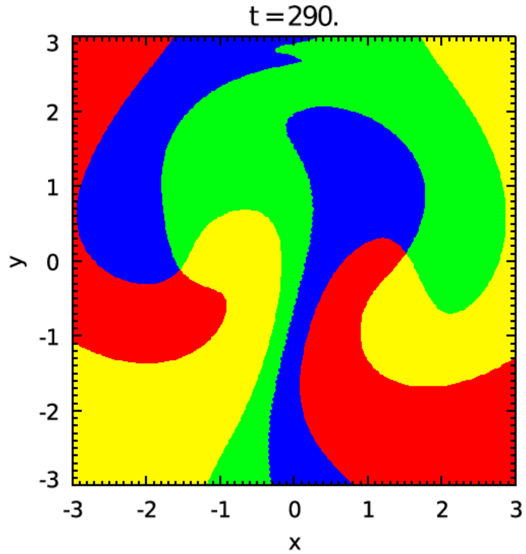
$t_i = +1$

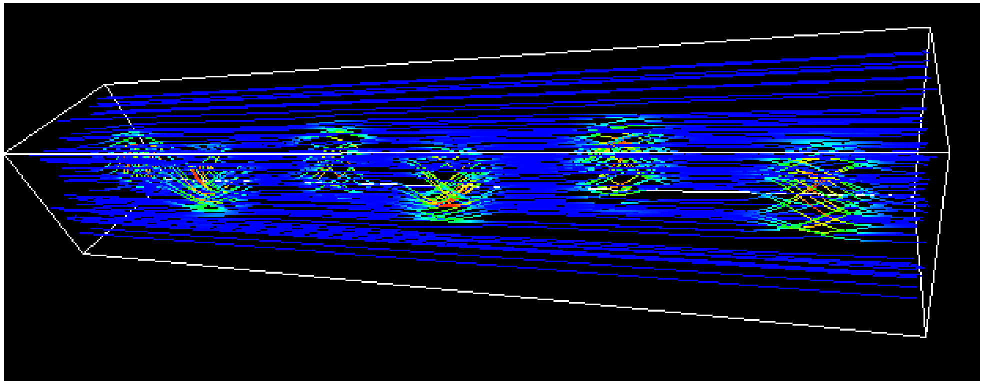


$t_i = -1$

Fixed point index: $T = \sum_i t_i$ conserved for $\lim \eta \to 0$



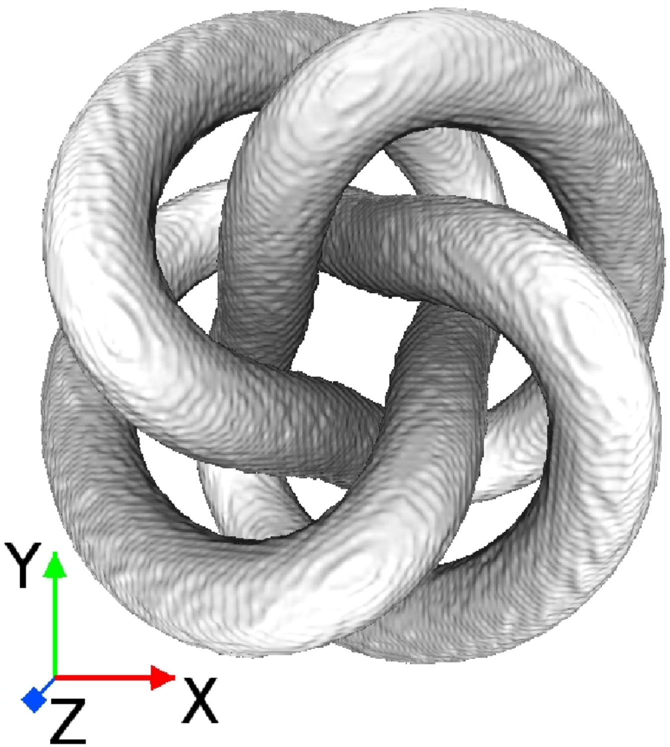




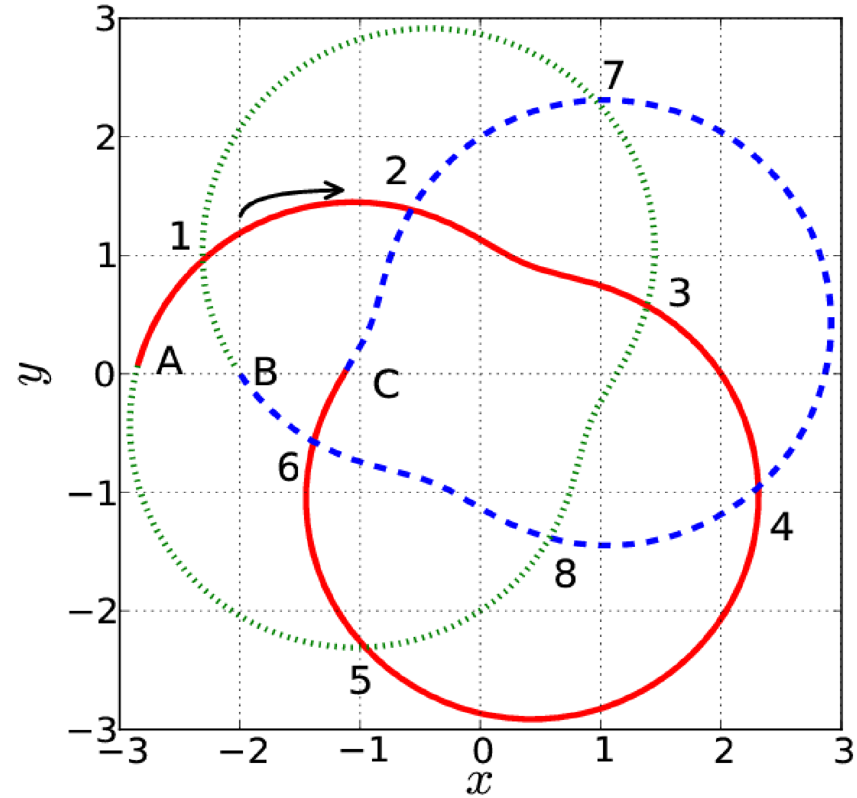
Taylor state is not reached
$\rightarrow$ T is additional constraint

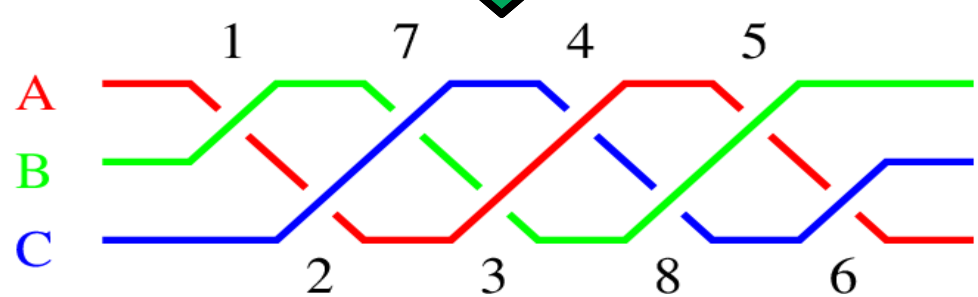
Braid Representation

need $B_z > 0$ ➡ braid representation of knots and links



4-foil knot

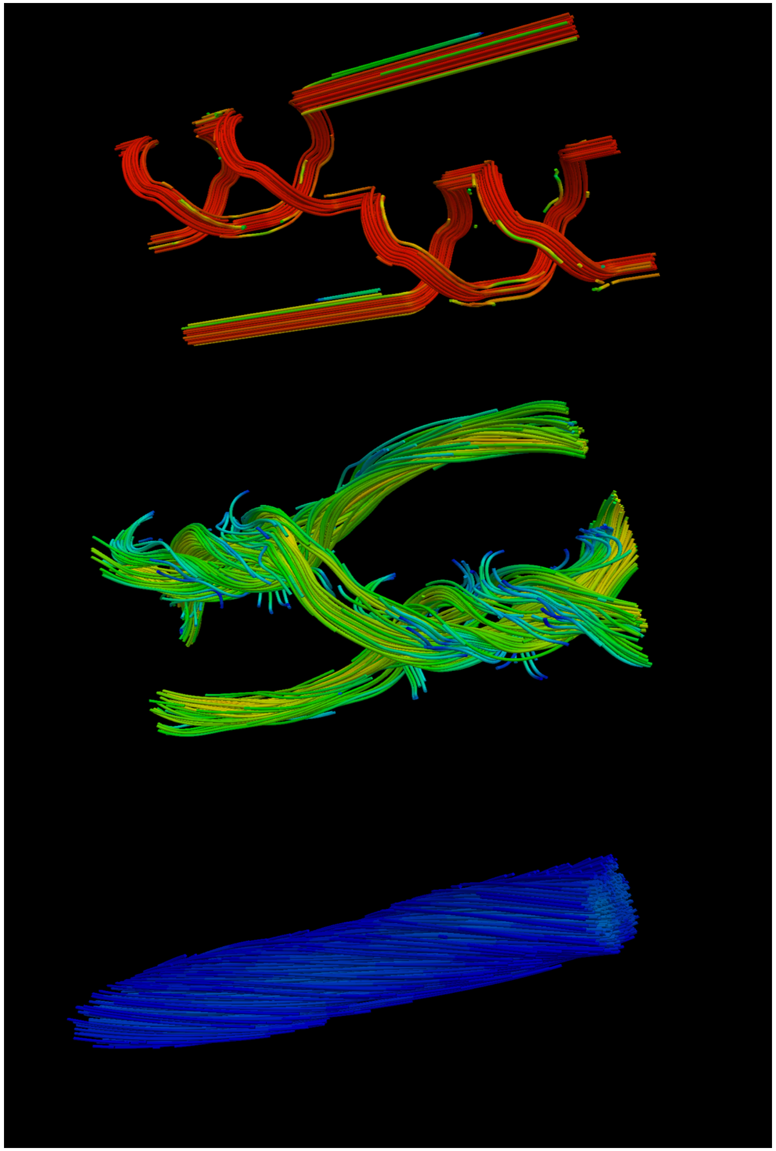




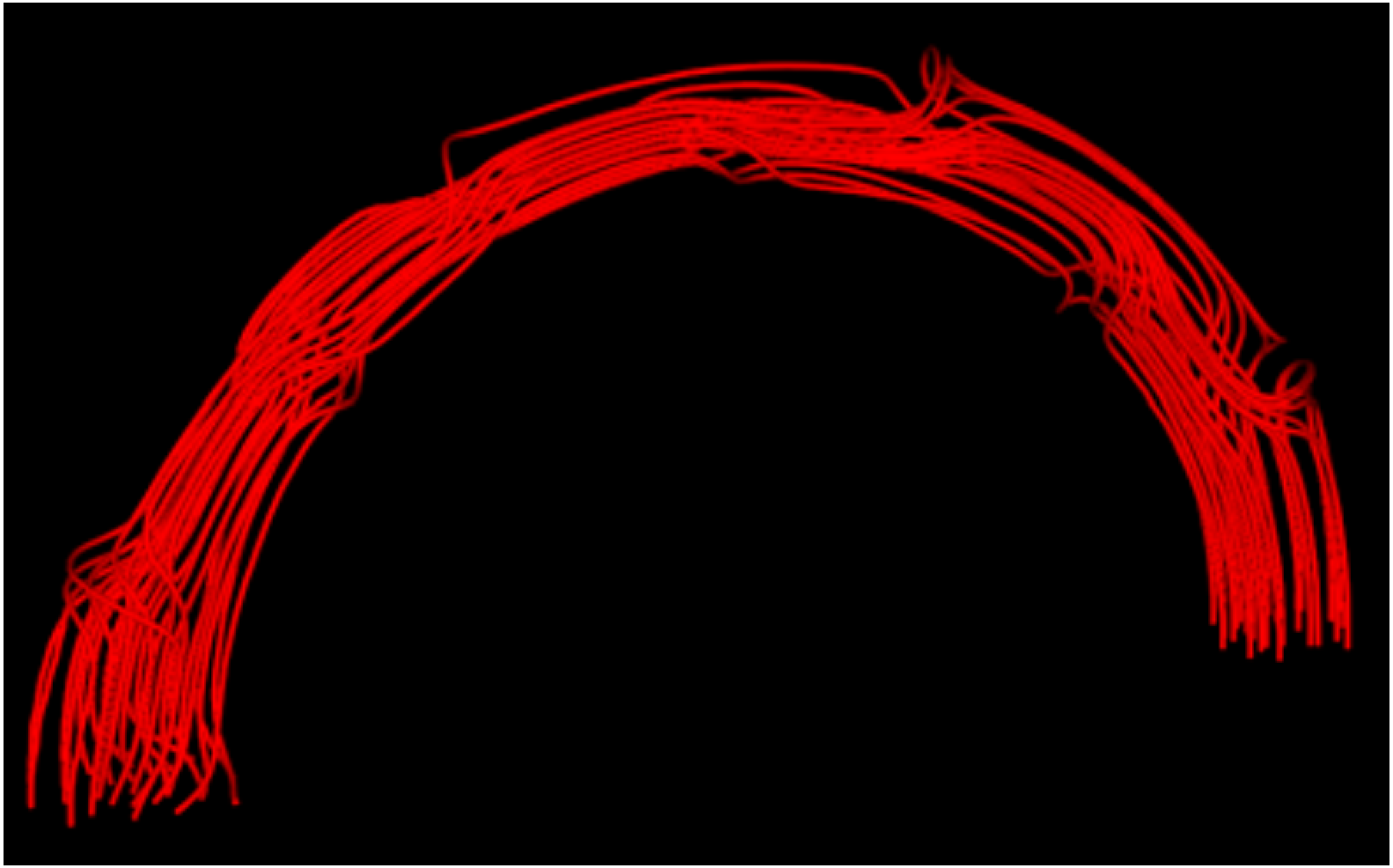
Magnetic Braid Configurations

AAA (trefoil knot)

AABB (Borromean rings)



Magnetic Braid Configurations

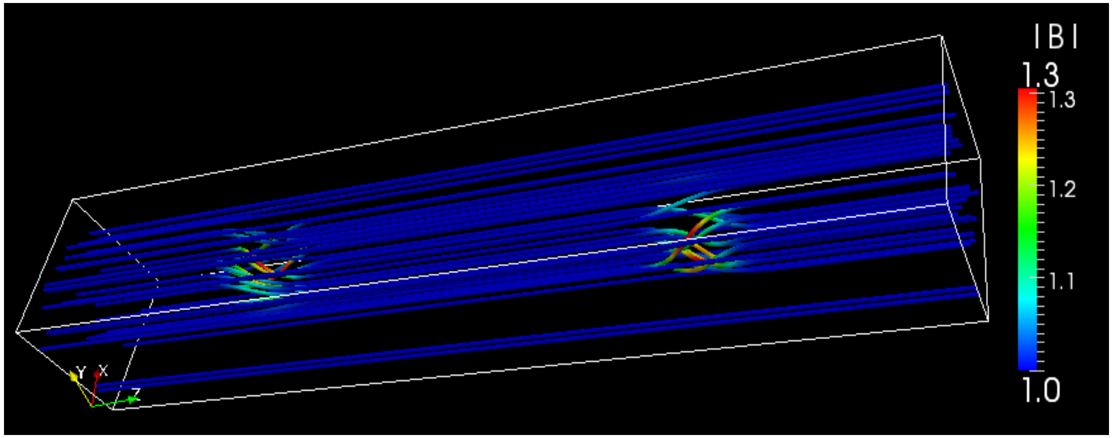


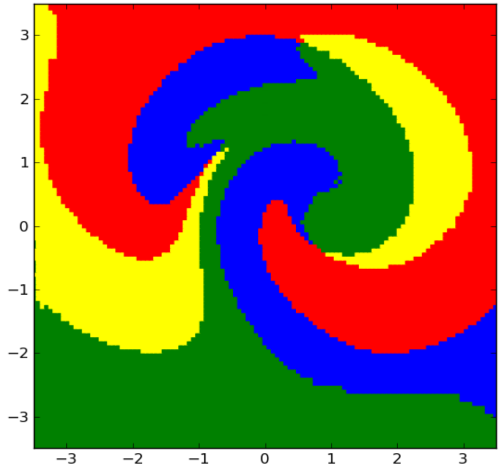
(Prior and MacTaggart 2016)

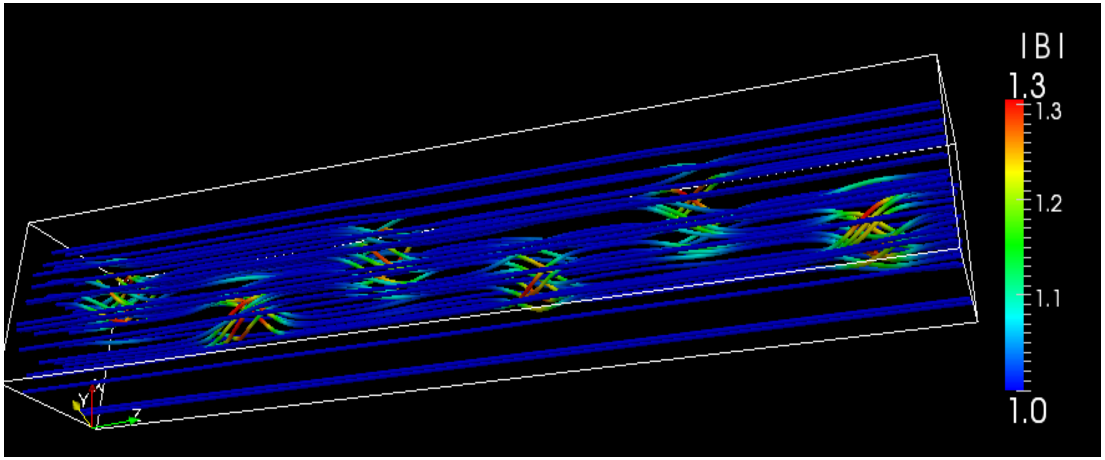
Conclusions

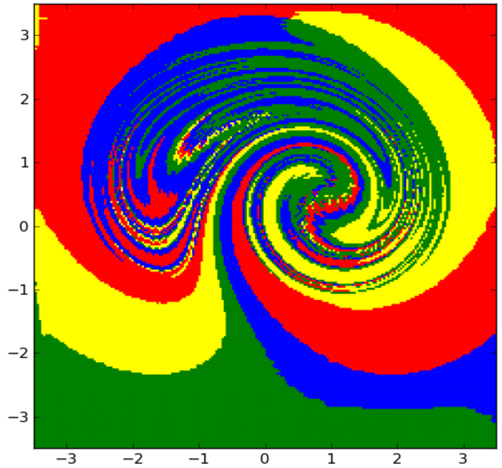
- Magnetic helicity as constraint on plasma dynamics.
- Further topological constraint: fixed point index, field line helicity, quadratic helicities.
- Topology preserving relaxation of magnetic fields.
- Other topological invariants? Jones polynomials?

Field Line Tracing









Generalized flux function:

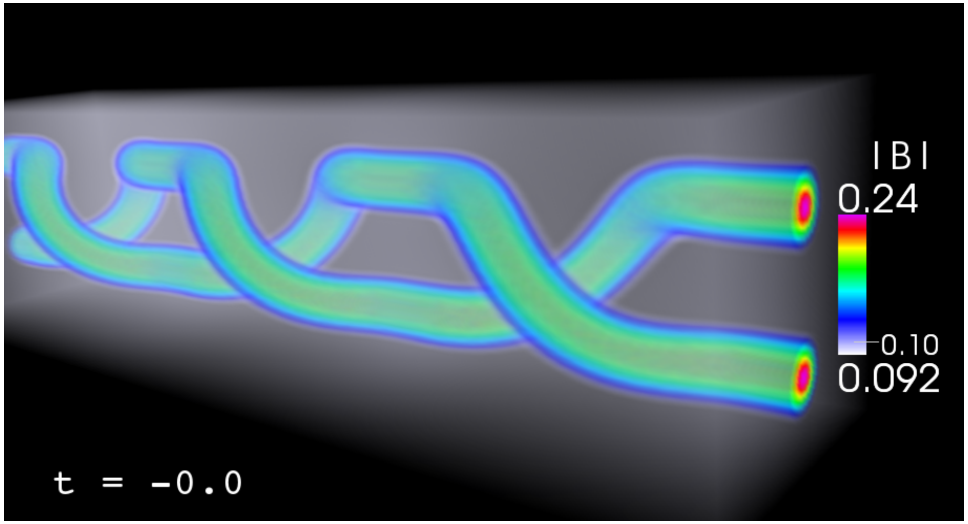
$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

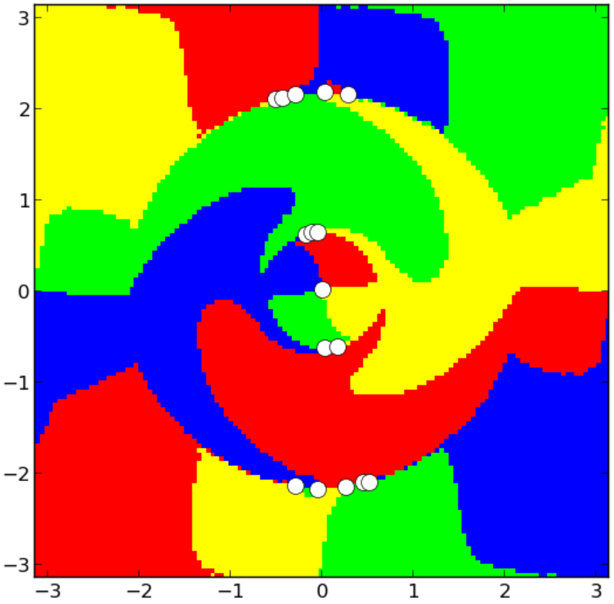
Reconnection rate:

$$\sum_i \frac{\mathrm{d}\mathcal{A}(\mathbf{x}_i)}{\mathrm{d}t}$$

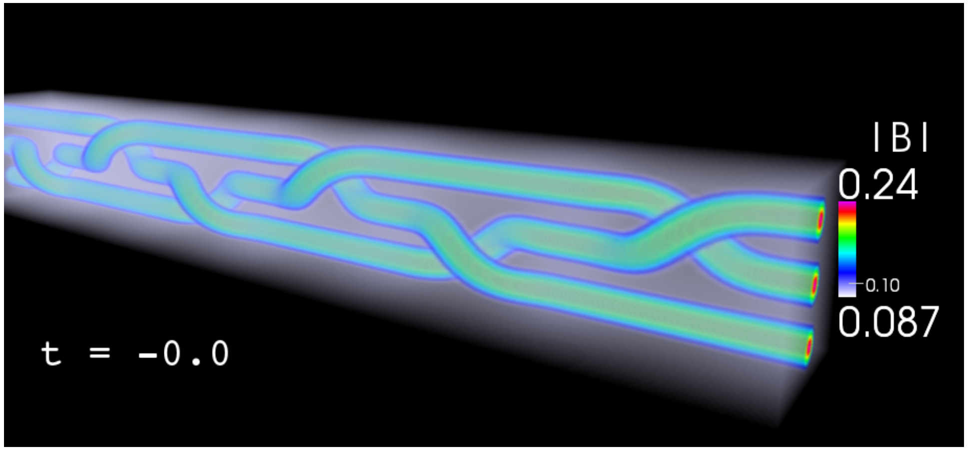
Knots as Braids

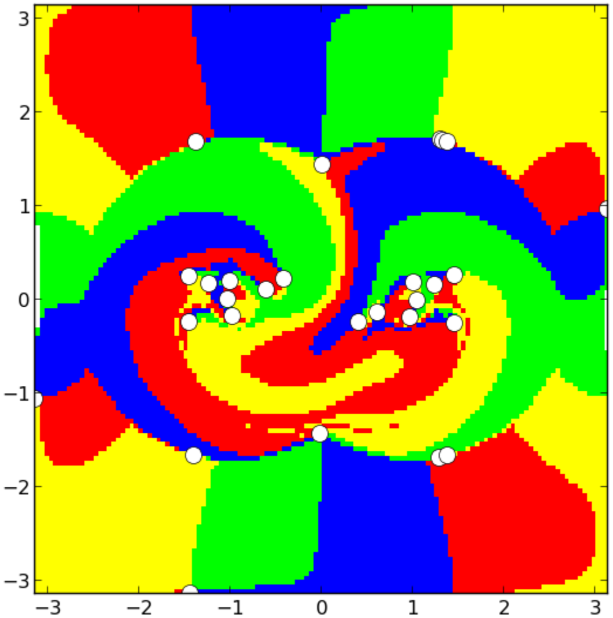
AAA, trefoil knot





AbAbAb, Borromean rings





Helical Dynamos

kinetic helicity $\boldsymbol{\omega} \cdot \boldsymbol{u}$ → helical magnetic fields $\overline{\boldsymbol{a} \cdot \boldsymbol{b}}$ ↑ $\overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{B}}$ ↓

α effect → growth of large-scale fields