Lagrangian Relaxation of Magnetic Fields

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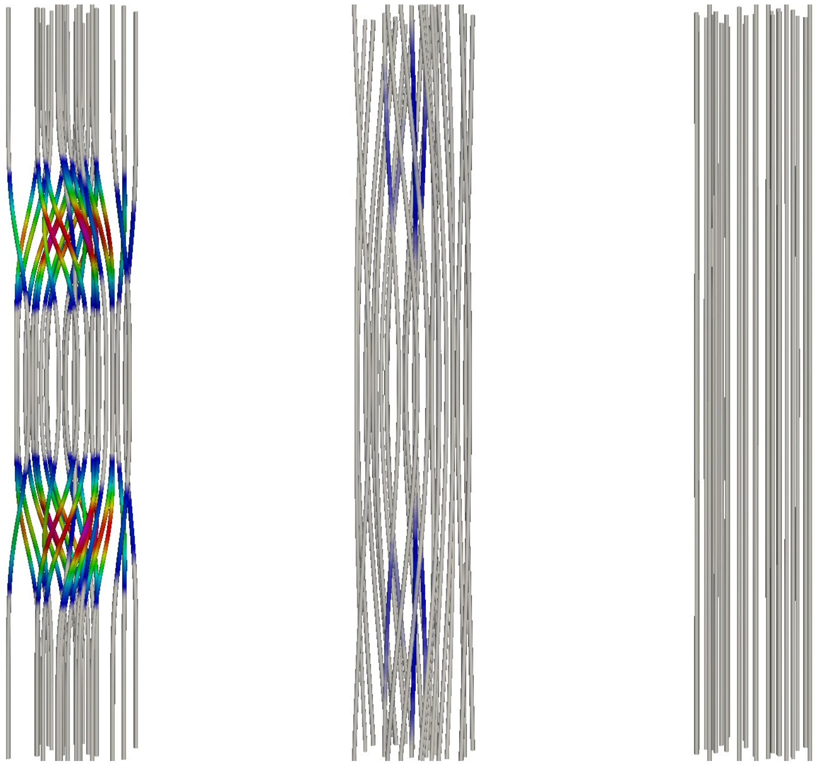


Force-Free Magnetic Fields

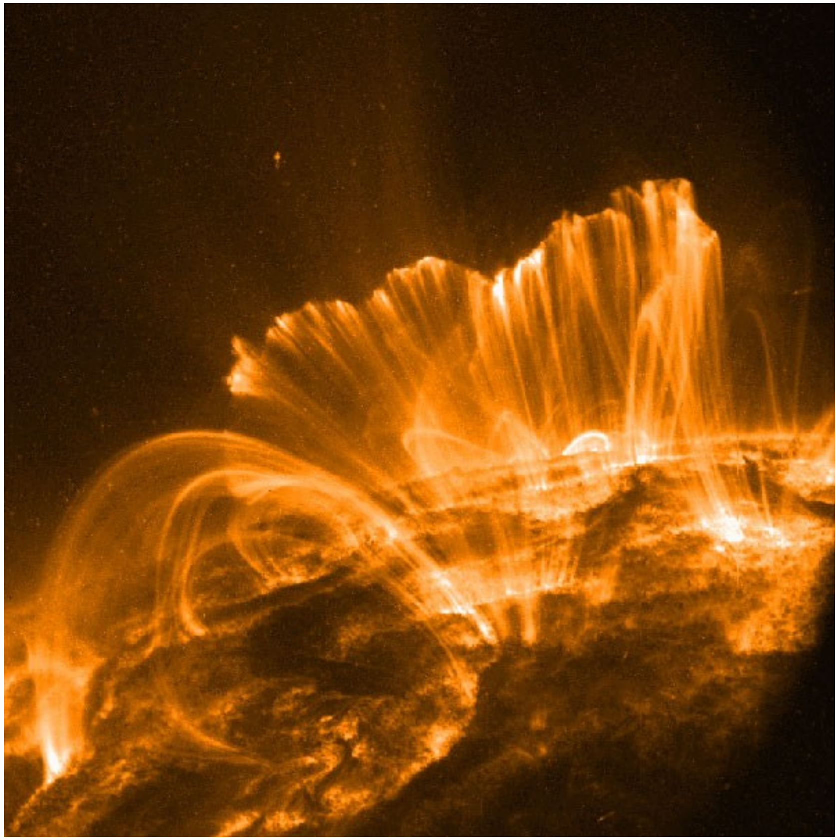
Solar corona: low plasma beta and magnetic resistivity

- ➡ Force-free magnetic fields
- ➡ Minimum energy state

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \;\Leftrightarrow\; \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

Beltrami field



NASA

Problem:
Find a force-free state for a magnetic field with given topology.

Here:
Numerical method for finding such states.

Ideal Field Relaxation

Ideal induction eq.: $$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

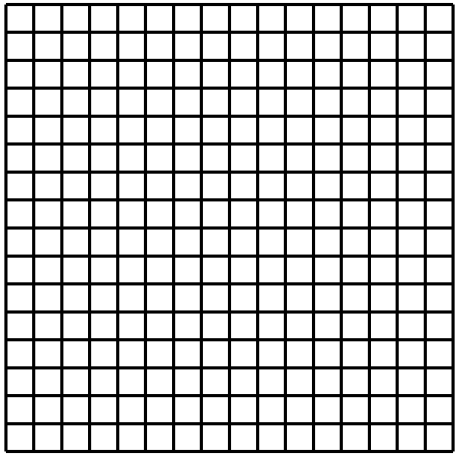
Frozen in magnetic field.

(Batchelor, 1950)

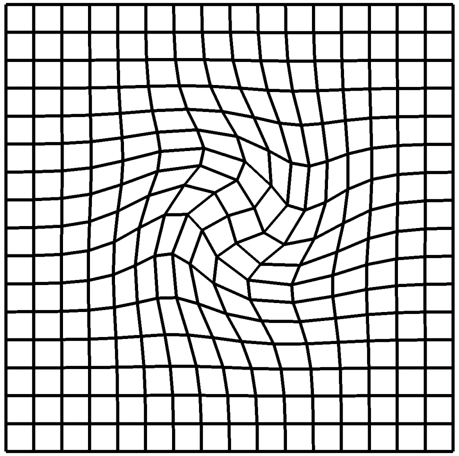
But: Numerical diffusion in finite difference Eulerian codes.

Solution: Lagrangian description of moving fluid particles:

$$\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$$



$$\mathbf{x}(\mathbf{X}, t)$$



Ideal Field Relaxation

Field evolution: $$B_i(\mathbf{X},t) = \frac{1}{\Delta}\sum_{j=1}^{3}\frac{\partial x_i}{\partial X_j}B_j(\mathbf{X},0)$$

$$\Delta = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

Preserves topology and divergence-freeness.

Grid evolution: $$\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \mathbf{u}\left(\mathbf{x}(\mathbf{X},t),t\right)$$

Magneto-frictional term: $\mathbf{u} = \gamma \mathbf{J} \times \mathbf{B} \qquad \mathbf{J} = \nabla \times \mathbf{B}$

$$\Rightarrow \frac{\mathrm{d}E_{\mathrm{M}}}{\mathrm{d}t} < 0$$

(Craig and Sneyd 1986)

Numerical Curl Operator

Compute $\mathbf{J} = \nabla \times \mathbf{B}$ on a distorted grid:

$$\frac{\partial B_i}{\partial x_j} = X_{\alpha,j}(x_{i,\alpha\beta} B^0_\beta \Delta^{-1} + x_{i,\beta} B^0_{\beta,\alpha} \Delta^{-1} - x_{i,\beta} B^0_\beta \Delta^{-2} \Delta_{,\alpha})$$

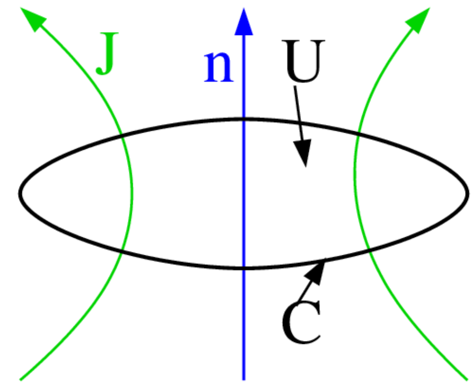
(Craig and Sneyd 1986)

$$B^0_i = B_i(0)$$

- Multiplication of several terms leads to high numerical errors.
- Current not divergence free: $\nabla \cdot \mathbf{J} \neq 0$
- Only reaching a certain force-freeness. *(Pontin et al. 2009)*

Mimetic Numerical Operators

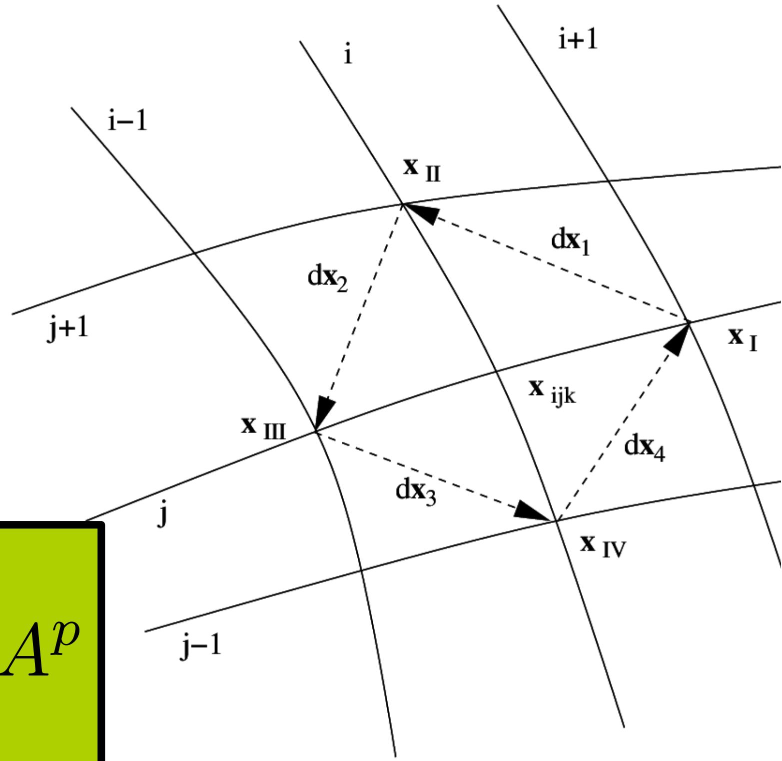
$$I = \int_U \mathbf{J} \cdot \mathbf{n}\, \mathrm{d}S = \oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r}$$



Discretized:

$$I \approx \mathbf{J}(\mathbf{X}_{ijk}) \cdot \mathbf{n} A = \sum_{r=1}^{4} \mathbf{B}_r \cdot \mathrm{d}\mathbf{x}_r$$

$$\mathbf{J}(\mathbf{X}_U) \approx \mathbf{J}(\mathbf{X}_{\mathbf{ijk}}), \quad \mathbf{X}_U \in U$$



3 planes will give 3 l.i. normal vectors:

$$I^p = \mathbf{J}(\mathbf{X}_{ijk}) \cdot \mathbf{n}^p = \sum_{r=1}^{4} \mathbf{B}_r^p \cdot \mathrm{d}\mathbf{x}_r / A^p$$

Inversion yields $\mathbf{J}$ with $\nabla \cdot \mathbf{J} = 0$.

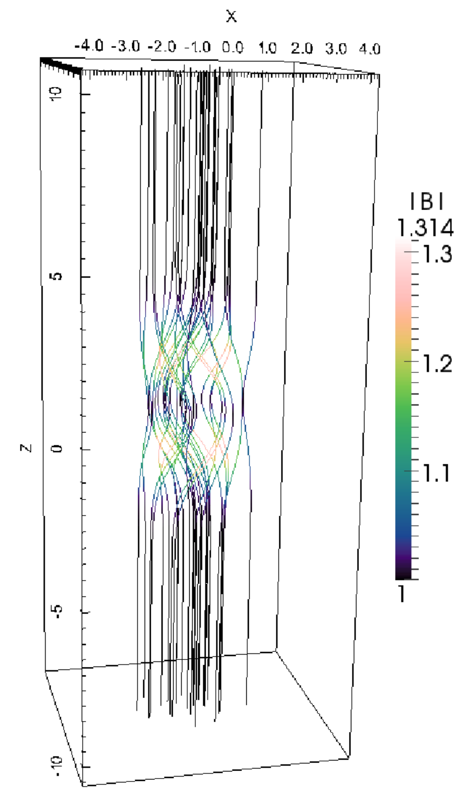
(Hyman, Shashkov 1997)

Simulations

- GPU code GLEMuR (**G**pu-based **L**agrangian mim**E**tic **M**agnetic **R**elaxation)
- line tied boundaries
- mimetic vs. classic

(Candelaresi et al. 2014)

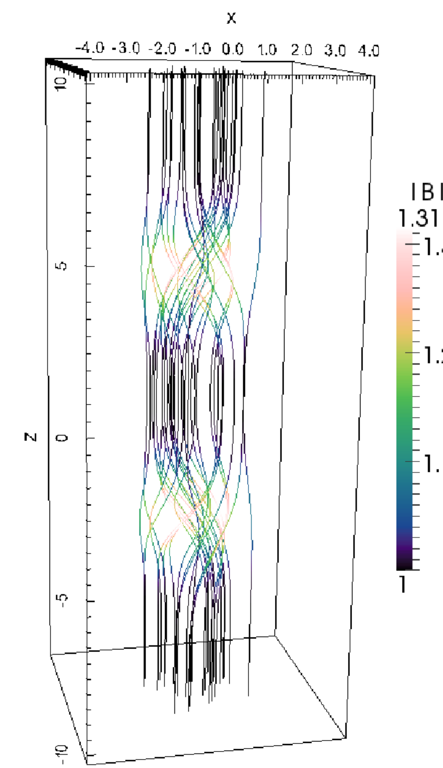
Nvidia Tesla K40



we know:

$$\lim_{t\to\infty} \mathbf{B}(t)$$

$$\lim_{t\to\infty} \mathbf{x}(t)$$



we know:

$$\lim_{t\to\infty} \mathbf{B}(t)$$

Quality Parameters

Deviation from the expected relaxed state:

$$\sigma_{\mathbf{x}} = \sqrt{\frac{1}{N} \sum_{ijk} (\mathbf{x}(\mathbf{X}_{ijk}) - \mathbf{x}_{\mathrm{relax}}(\mathbf{X}_{ijk}))^2}$$

$$\sigma_{\mathbf{B}} = \sqrt{\frac{1}{N} \sum_{ijk} (\mathbf{B}(\mathbf{X}_{ijk}) - \mathbf{B}_{\mathrm{relax}}(\mathbf{X}_{ijk}))^2}$$

Free magnetic energy:

$$E_{\mathrm{M}}^{\mathrm{free}} = E_{\mathrm{M}} - E_{\mathrm{M}}^{\mathrm{bkg}}$$

$$E_{\mathrm{M}} = \int_V \mathbf{B}^2/2 \, \mathrm{d}V \qquad \mathbf{B}^{\mathrm{bkg}} = B_0 \hat{e}_z$$

Quality Parameters

For a force-free field: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

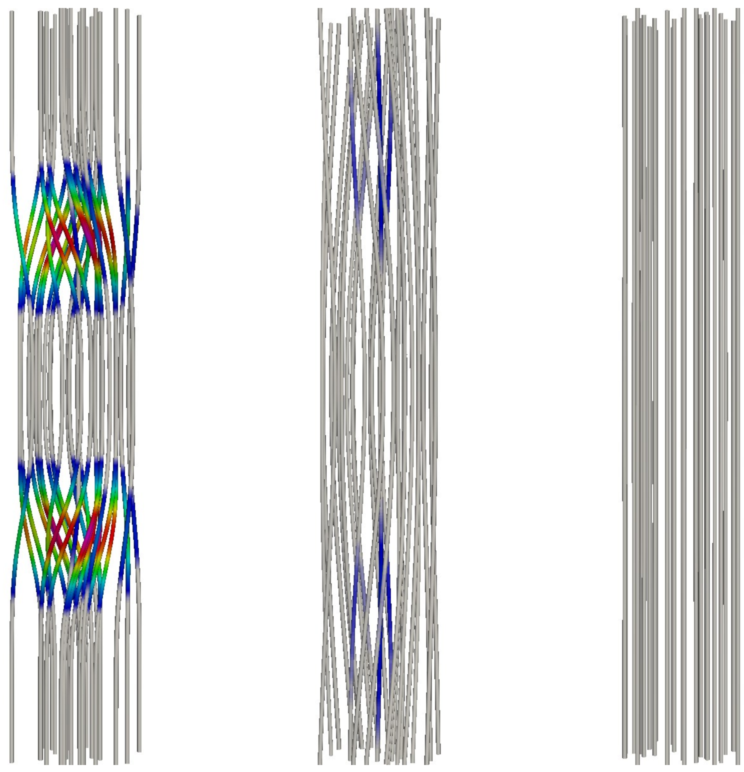
- Force-free parameter does not change along field lines.
- Measure the change of $\alpha^* = \dfrac{\mathbf{J} \cdot \mathbf{B}}{\mathbf{B}^2}$ along field lines:

$$\epsilon^* = \max_{i,j} \left(a_r \frac{\alpha^*(\mathbf{X}_i) - \alpha^*(\mathbf{X}_j)}{|\mathbf{X}_i - \mathbf{X}_j|} \right); \quad \mathbf{X}_i, \mathbf{X}_j \in s_\alpha$$

Particular field line: $s_\alpha = \{(0,0,Z) : Z \in [-L_z/2, L_z/2]\}$

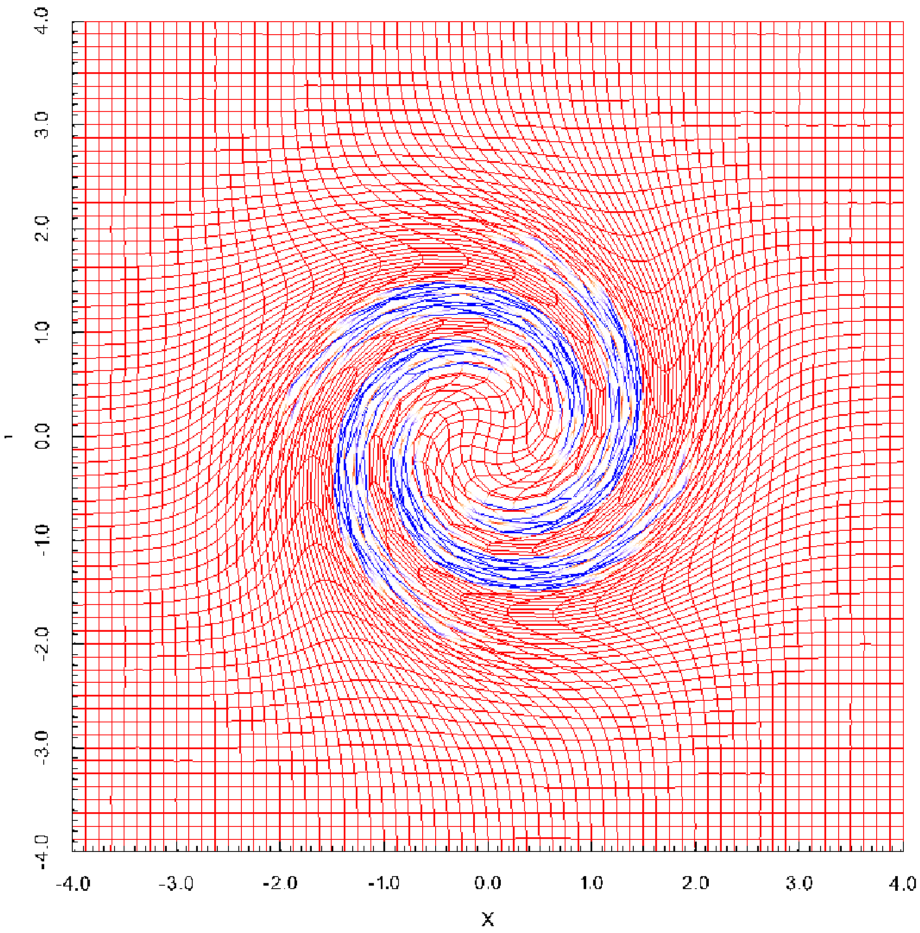
Field Relaxation

Magnetic streamlines:



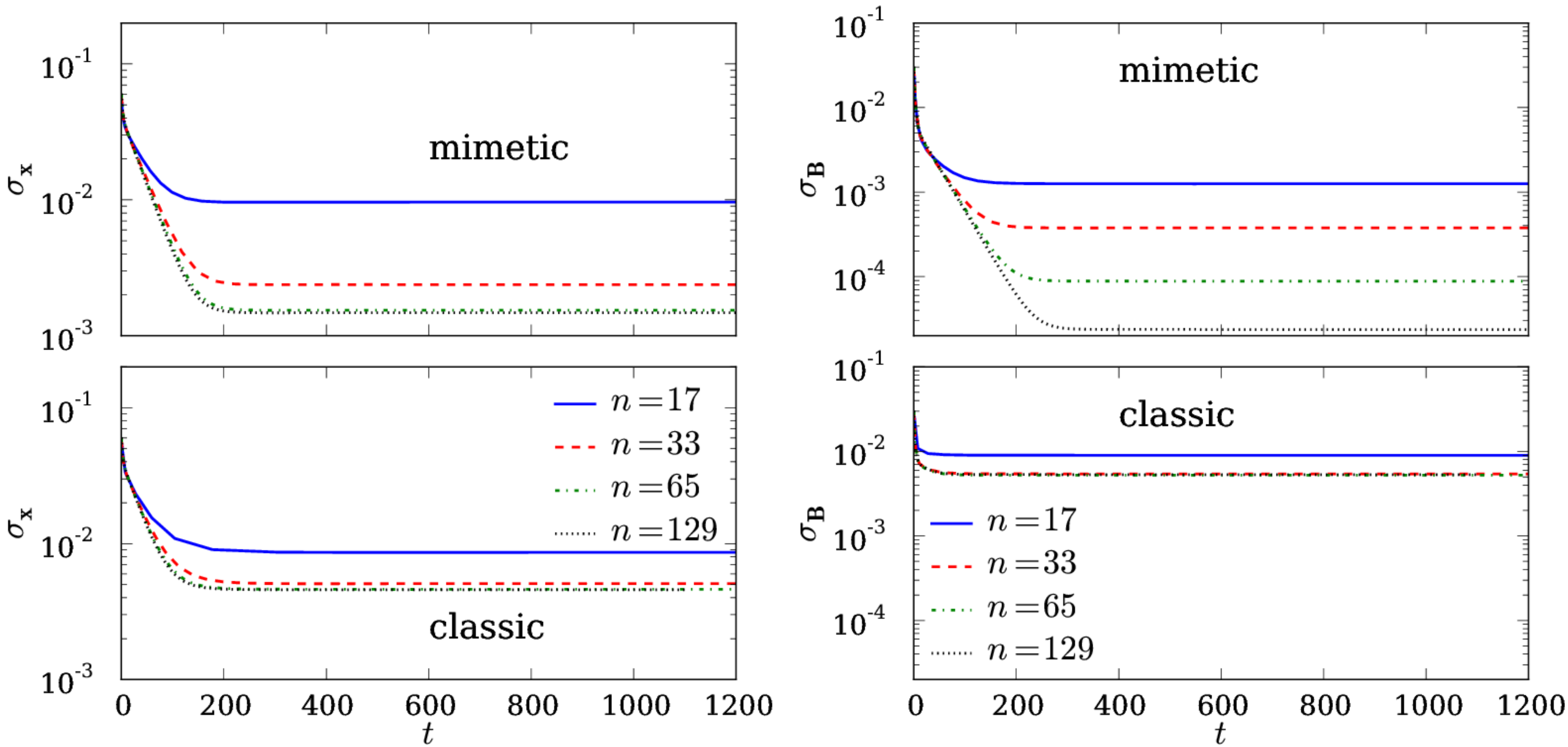
$t = 0$ $t = 10$ $t = 430$

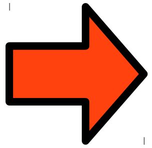
Grid distortion at mid-plane:



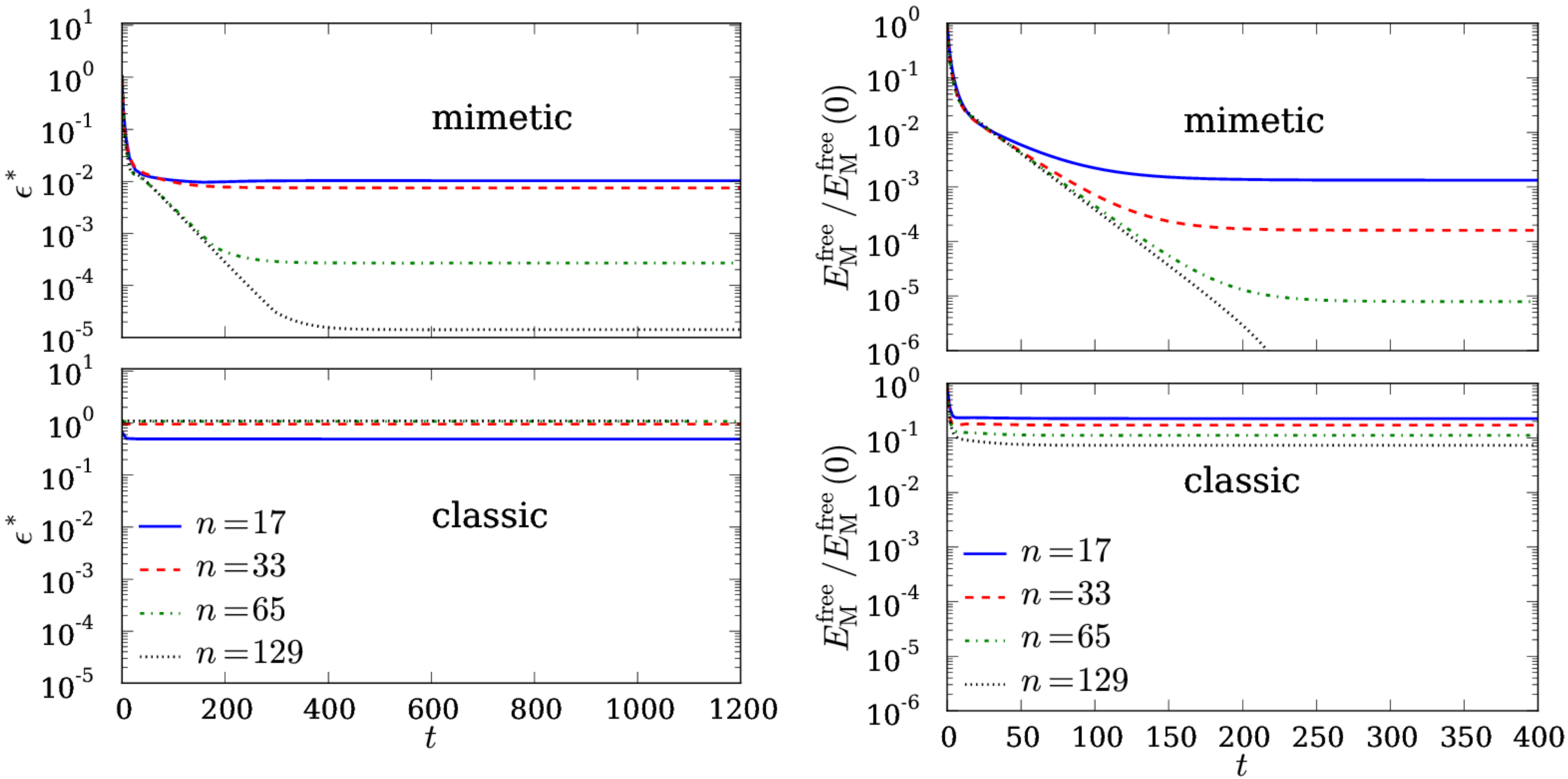
movie

Relaxation Quality




Closer to the analytical solution by 3 orders of magnitude.

Relaxation Quality



Closer to force-free state by 5 orders of magnitude.

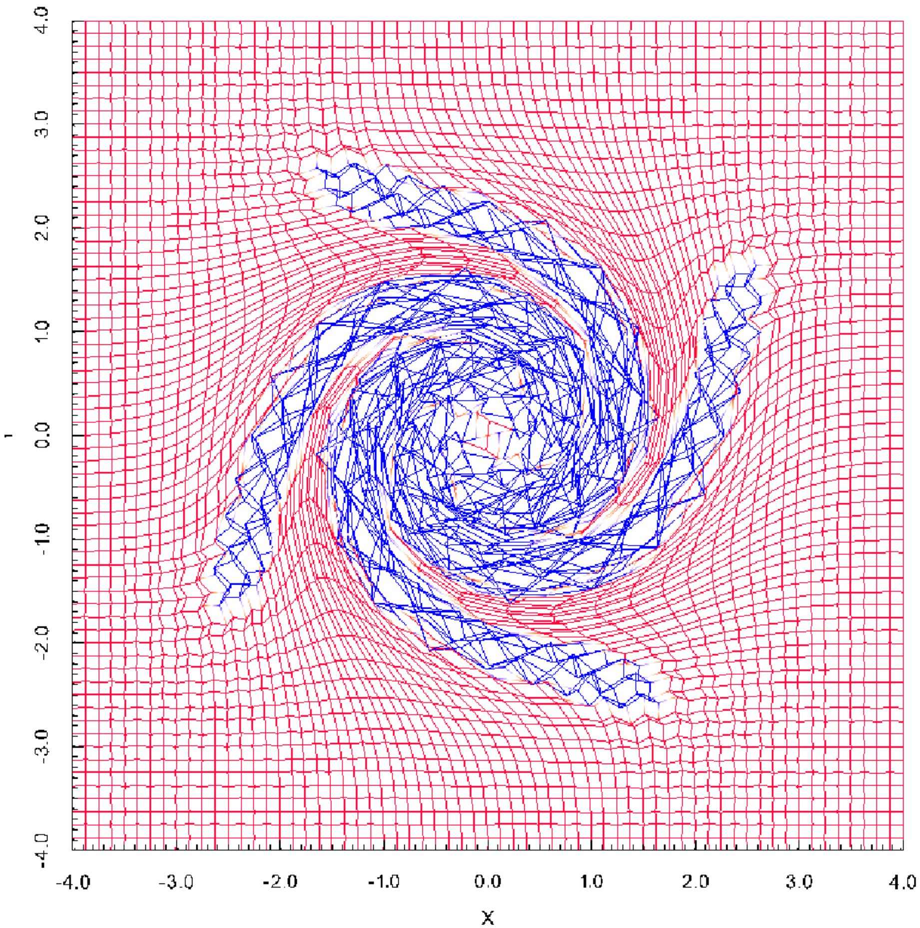
Performance Gain

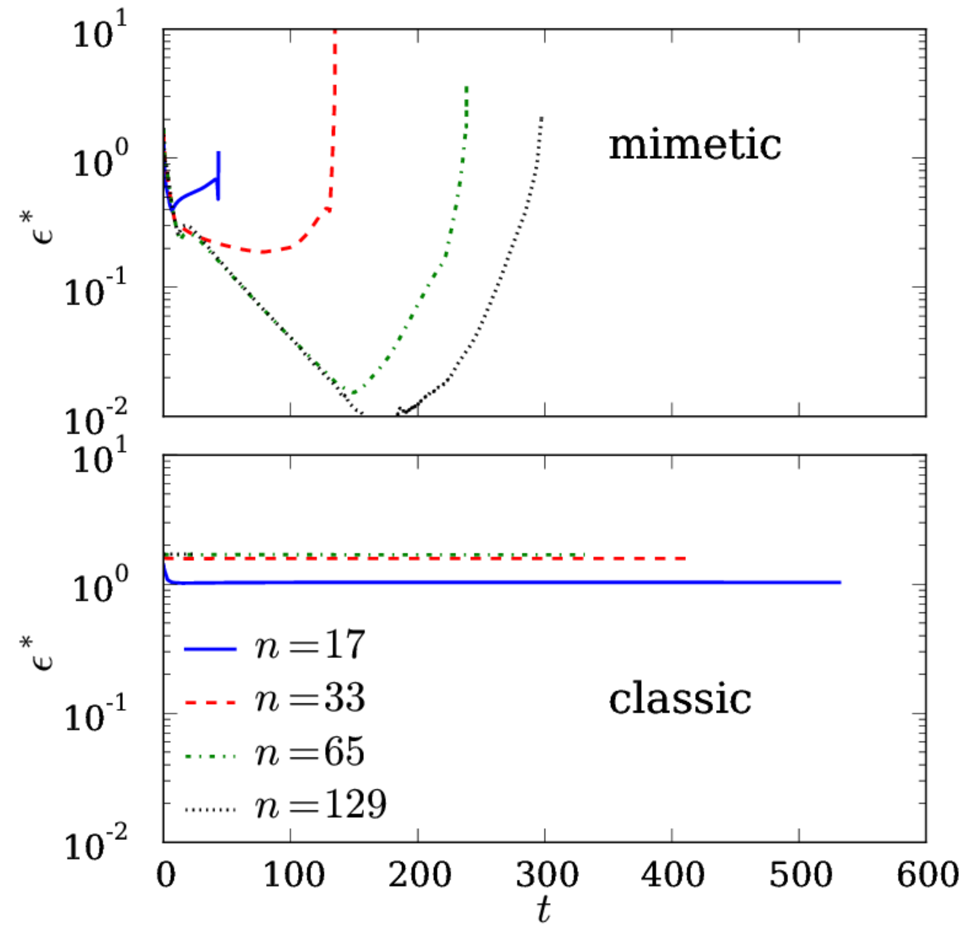
	mimetic vs. classic
floating point operations	1/2
computation time (gross)	1/2
previous code*	x100

*serial code using classical finite differences and an implicit solver

(Craig and Sneyd 1986)

Limitations





red: convex
blue: concave

For concave cells the method becomes unstable.
But: results before crash better than classic method.

Conclusions

- Lagrangian numerical scheme for ideal evolution.
- Preserving field line topology.
- Mimetic methods more capable of producing force-free fields.
- GLEMuR code running on GPUs.
- Performance gain of x2 compared to classical approach.

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