Helicity Fluxes in Dependence of the Gauge

Simon Candelaresi

Division of Mathematics, University of Dundee, Dundee DD1 4HN, United Kingdom

1 Lie-Transport and Advective Gauge

In ideal, i.e. non-resistive, MHD we can write the induction equation as

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{1}$$

or using the vector potential $\boldsymbol{A}$ as

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \times \boldsymbol{B} + \boldsymbol{\nabla} \frac{\partial \Lambda}{\partial t}, \tag{2}$$

with the gauge Λ that we are free to choose from the set of differentiable functions $\mathbb{R}^4 \to \mathbb{R}$ satisfying the boundary conditions.

From differential geometry we know about the Lie-transport of differentiable forms that are associated to vector fields. To the vector potential $\boldsymbol{A}$ we can associate a 1-form, since $\boldsymbol{A}$ is a line density. Similarly, to the magnetic field $\boldsymbol{B}$ we can associate a differentiable 2-form, since it is a surface density. If $\boldsymbol{A}$ were idealy advected by a flow generated by the velocity field $\boldsymbol{u}$ it would follow the equation

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \times \boldsymbol{B} - \boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{A}). \tag{3}$$

Combining equations (2) and (3) the gauge field must follow the evolution equation

$$\frac{\partial \Lambda}{\partial t} = -\boldsymbol{u} \cdot \boldsymbol{A}. \tag{4}$$

If we started with a vector potential $\boldsymbol{A}$ in some gauge, we evolve $\boldsymbol{A}$ according to equation (2) and the gauge field Λ according to equation (4). However, this requires an initial condition for Λ and it's boundary condition to be defined. But, since $\boldsymbol{A}$ is now Lie-transported, we should not need to solve for an evolution equation for $\boldsymbol{A}$ any more as long as we know the fluid's position at any time and can differentiate according to their initial positions. However, this is in general not possible due to strong turbulent motions.

This gauge has the property that we can easily calculate the evolution of the magnetic helicity density as its advective flux:

$$\begin{aligned} \frac{\partial h}{\partial t} &= \frac{\partial \boldsymbol{A}}{\partial t} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{B}}{\partial t} && (5) \\ &= -\boldsymbol{\nabla} \cdot (\boldsymbol{u} h). && (6) \end{aligned}$$

This is just the Lie transport of a density h.

2 Advecto-Resistive Gauge

The above considerations are not new and have been used in the past (Candelaresi et al., 2011) for the resistive system. With this additional term equation (3) reads

$$\frac{\mathrm{D}A_i^{\mathrm{ar}}}{\mathrm{D}t} = -u_{j,i}A_j^{\mathrm{ar}} + \eta\boldsymbol{\nabla}^2 A_i^{\mathrm{ar}}. \tag{7}$$

Numerical experiments have shown (Candelaresi et al., 2011) that solving this equation directly leads to numerical instabilities. Therefore, it is preferred to solve the induction equation in the resistive gauge

$$\frac{\partial \boldsymbol{A}^{\mathrm{r}}}{\partial t} = \boldsymbol{u}\times\boldsymbol{B} + \eta\boldsymbol{\nabla}^2\boldsymbol{A}^{\mathrm{r}}, \tag{8}$$

which is numerically well behaved. Hence, all their simulations are performed in the resistive gauge.

In order to obtain the magnetic vector potential in the advecto-resistive gauge they apply the gauge transformation

$$\boldsymbol{A}^{\mathrm{ar}} = \boldsymbol{A}^{\mathrm{r}} + \boldsymbol{\nabla}\Lambda^{\mathrm{r:ar}}, \tag{9}$$

with the gauge transformation $\Lambda^{\mathrm{r:ar}}$. Similarly to the ideal case, this field follows the evolution equation

$$\frac{\mathrm{D}\Lambda^{\mathrm{r:ar}}}{\mathrm{D}t} = -\boldsymbol{u}\cdot\boldsymbol{A}^{\mathrm{r}} + \eta\boldsymbol{\nabla}^2\Lambda^{\mathrm{r:ar}}. \tag{10}$$

With that the magnetic helicity density of the two gauges transform according to

$$h^{\mathrm{ar}} = h^{\mathrm{r}} + \boldsymbol{\nabla}\Lambda^{\mathrm{r:ar}}\cdot\boldsymbol{B}. \tag{11}$$

The helicity density follows

$$\frac{\partial h^{\mathrm{ar}}}{\partial t} = -2\eta\boldsymbol{J}\cdot\boldsymbol{B} - \boldsymbol{\nabla}\cdot\boldsymbol{F}^{\mathrm{ar}}, \tag{12}$$

with the advecto-resistive helicity flux

$$\boldsymbol{F}^{\mathrm{ar}} = h^{\mathrm{ar}}\boldsymbol{u} - \eta(\boldsymbol{\nabla}\cdot\boldsymbol{A}^{\mathrm{ar}})\boldsymbol{B} + \eta\boldsymbol{J}\times\boldsymbol{A}^{\mathrm{ar}}. \tag{13}$$

Contrast this to the resistive helicity flux

$$\boldsymbol{F}^{\mathrm{r}} = h^{\mathrm{r}}\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{A}^{\mathrm{r}} + \eta\boldsymbol{\nabla}\cdot\boldsymbol{A}^{\mathrm{r}})\boldsymbol{B} + \eta\boldsymbol{J}\times\boldsymbol{A}^{\mathrm{r}}. \tag{14}$$

3 Homogeneous Periodic Domain

Following Axel's turbulent simulations I perform the same simulations in the resistive gauge using the evolution equations for the gauge transformation (9) which I use to compute $\boldsymbol{A}^{\mathrm{ar}}$ from $\boldsymbol{A}^{\mathrm{r}}$, h^{ar} and $\boldsymbol{F}^{\mathrm{ar}}$. I solve the viscous and resistive induction equations in a periodic box for homogeneous helical forcing in a periodic box. The helicity spectra and fluxes are compared to the resistive gauge simulations, performed by Axel and are shown in Figure 1.

References

Candelaresi, S., Hubbard, A., Brandenburg, A., and Mitra, D. (2011). Magnetic helicity transport in the advective gauge family. *Phys. Plasmas*, 18:012903.

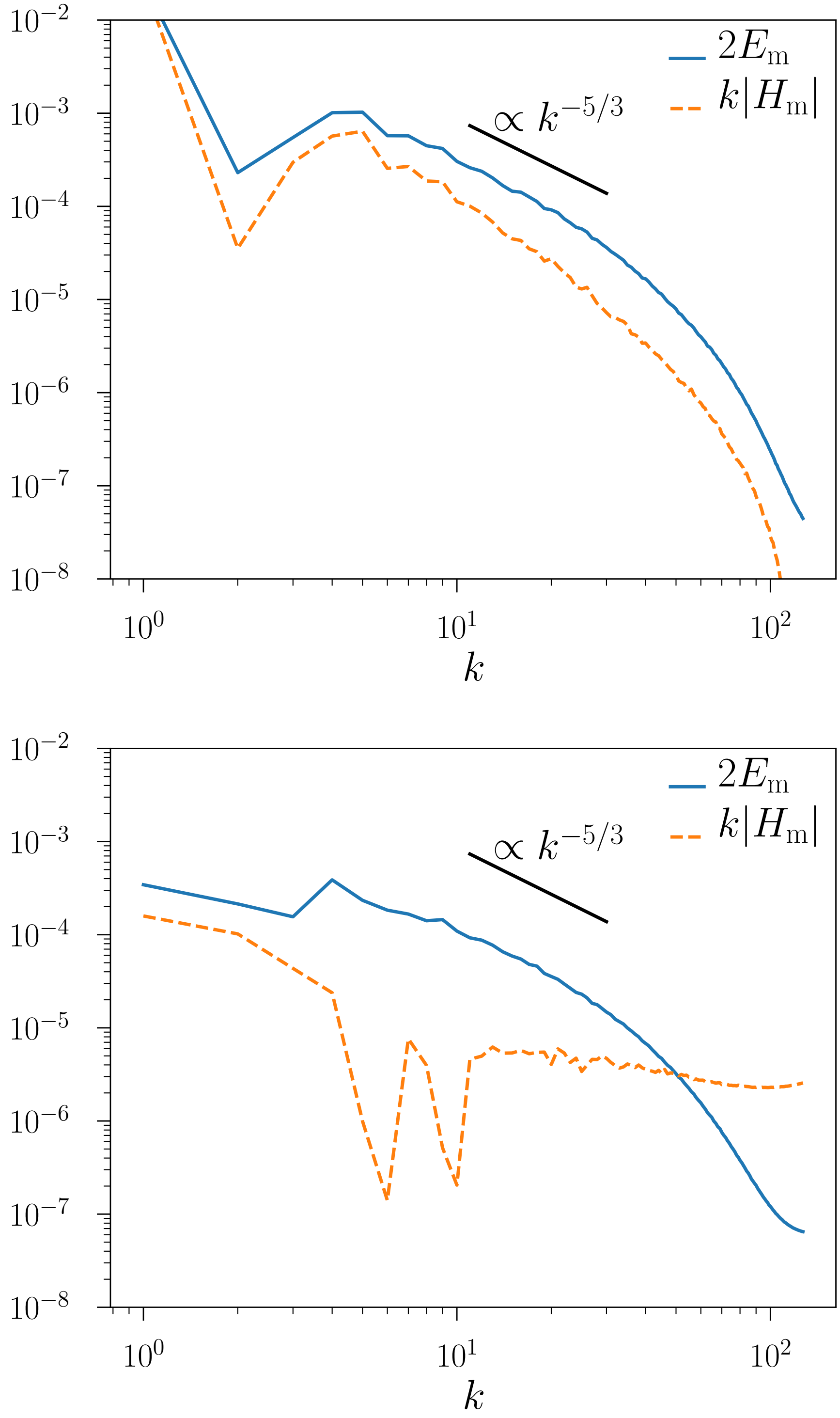

Figure 1: Compensated magnetic helicity spectra and magnetic energy spectra for the homogeneous turbulence case (upper panel) and inhomogeneous turbulence case (lower panel) at simulation time $t = 16000$.