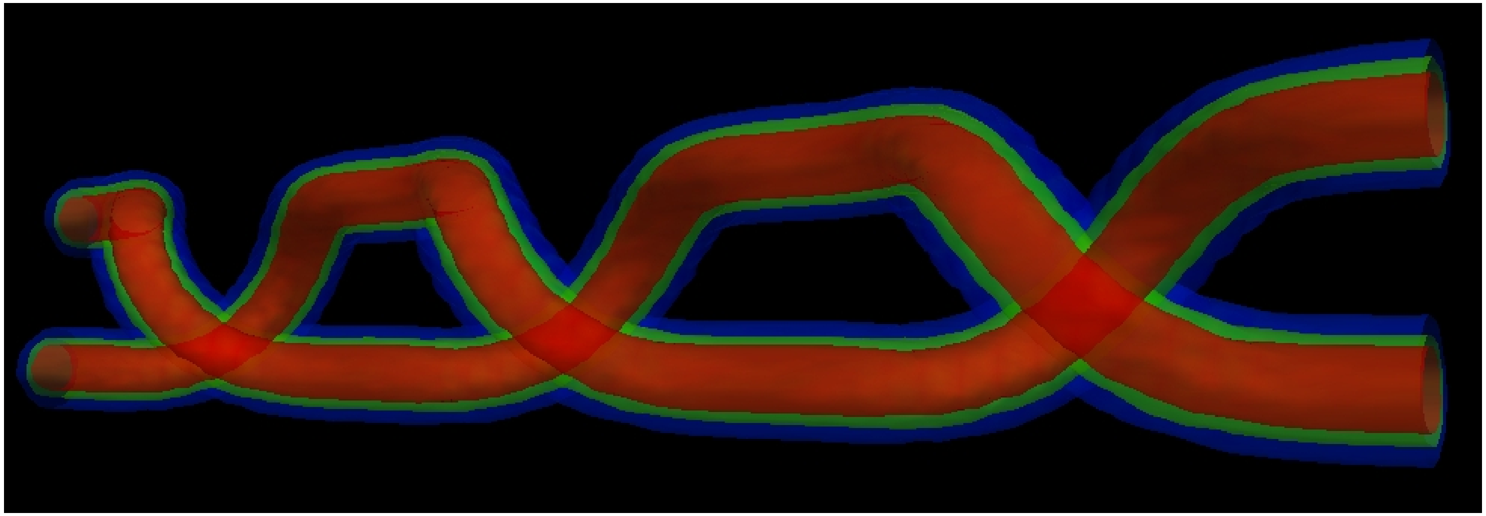
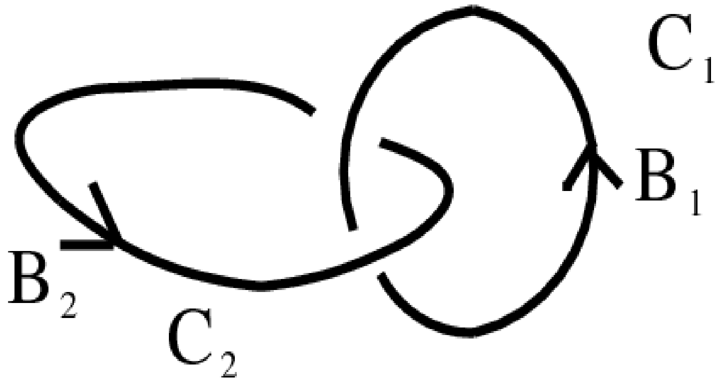
Topological constraints in magnetic field relaxation

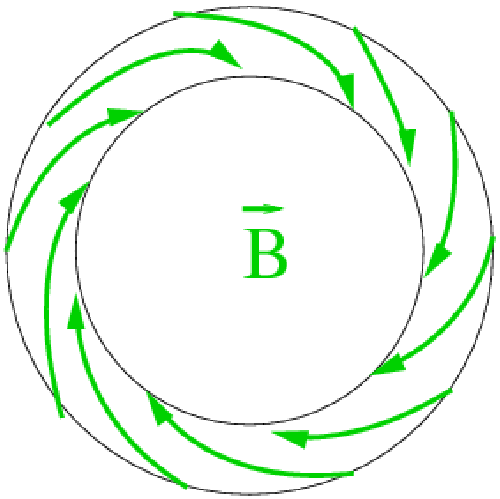
Simon Candelaresi



Topologies of Magnetic Fields

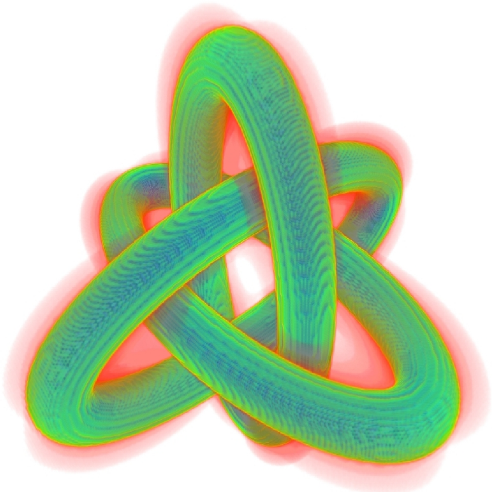


Hopf link

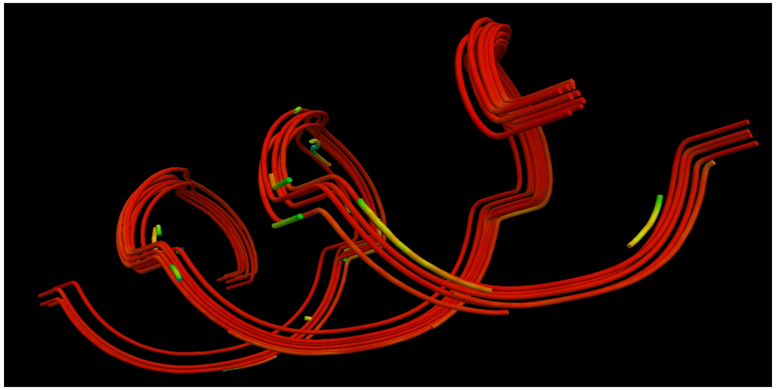


twisted field

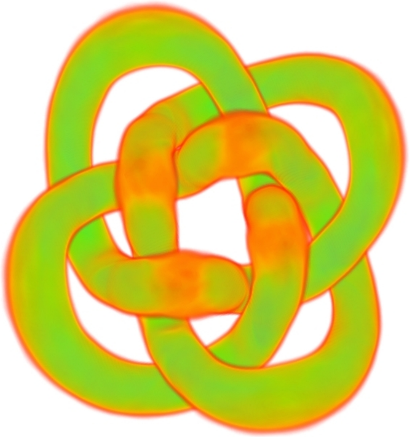
trefoil knot



Borromean rings



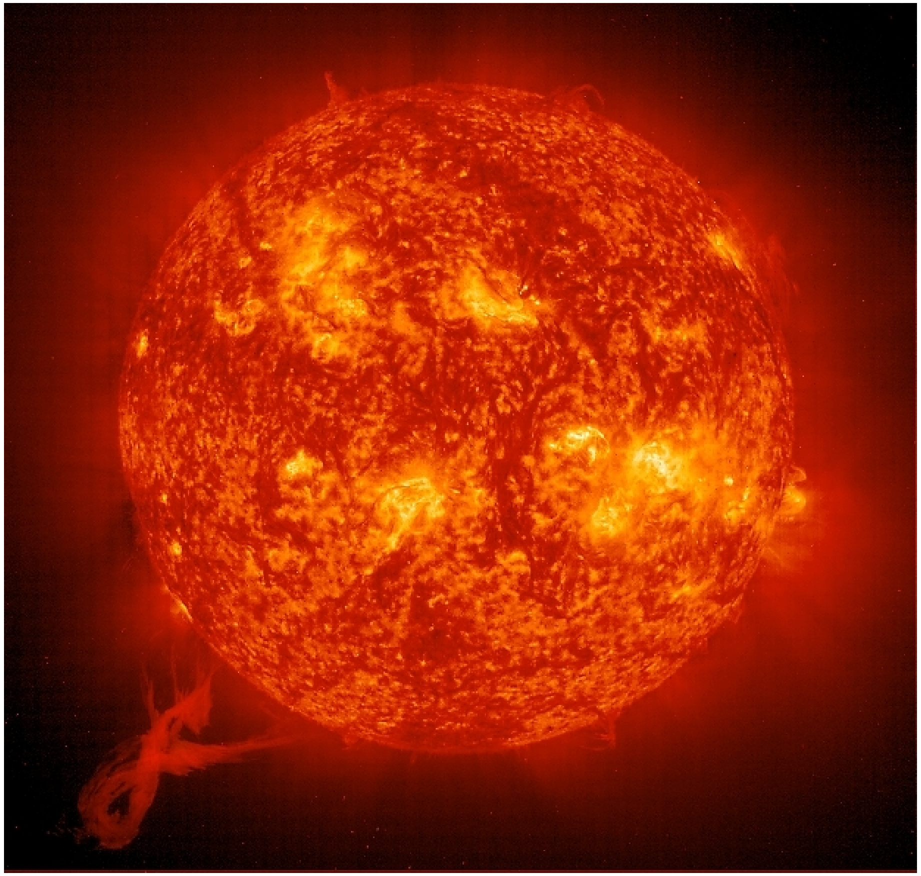
magnetic braid

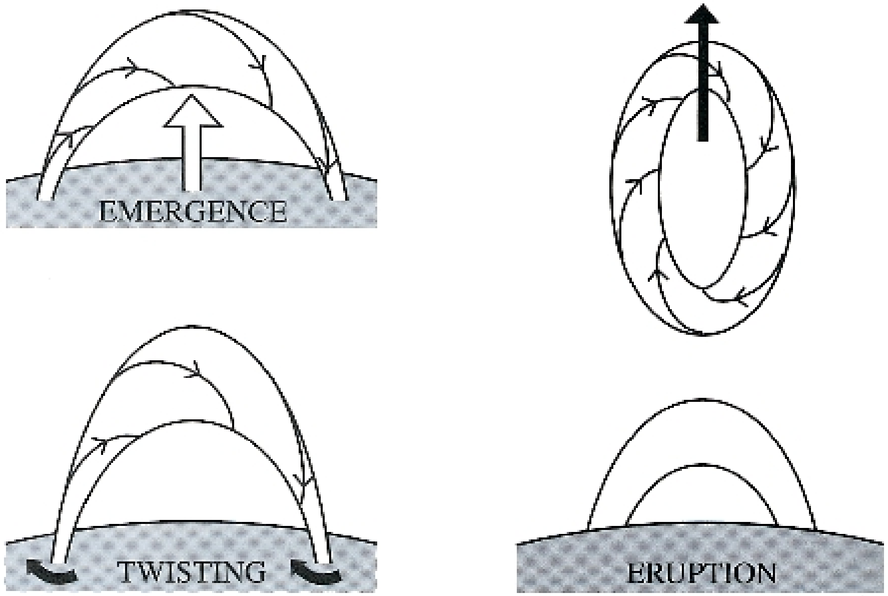


IUCAA knot

See **Ilan Roth's** poster for more knots and knot theory.

Twisted Magnetic Fields





Twisted fields are more likely to erupt *(Canfield et al. 1999)*.

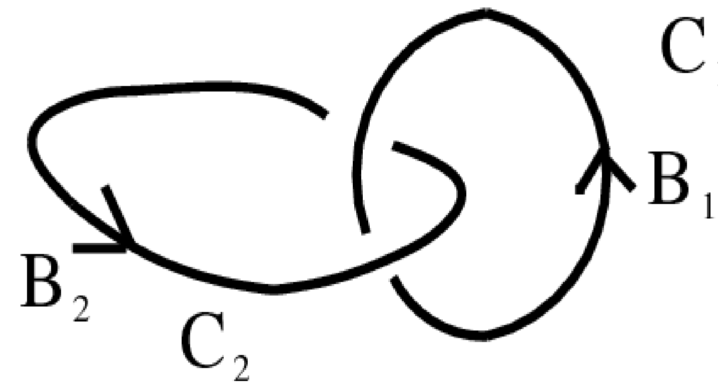
Tuesday's talk by **Zhang Mei**

Magnetic Helicity

Measure for the topology:

$$H_{\mathrm{M}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}V = 2n\phi_1\phi_2$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B} \qquad \phi_i = \int_{S_i} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$



$n =$ number of mutual linking

Conservation of magnetic helicity:

$$\lim_{\eta \to 0} \frac{\partial}{\partial t} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = 0 \qquad \eta = \text{magnetic resistivity}$$

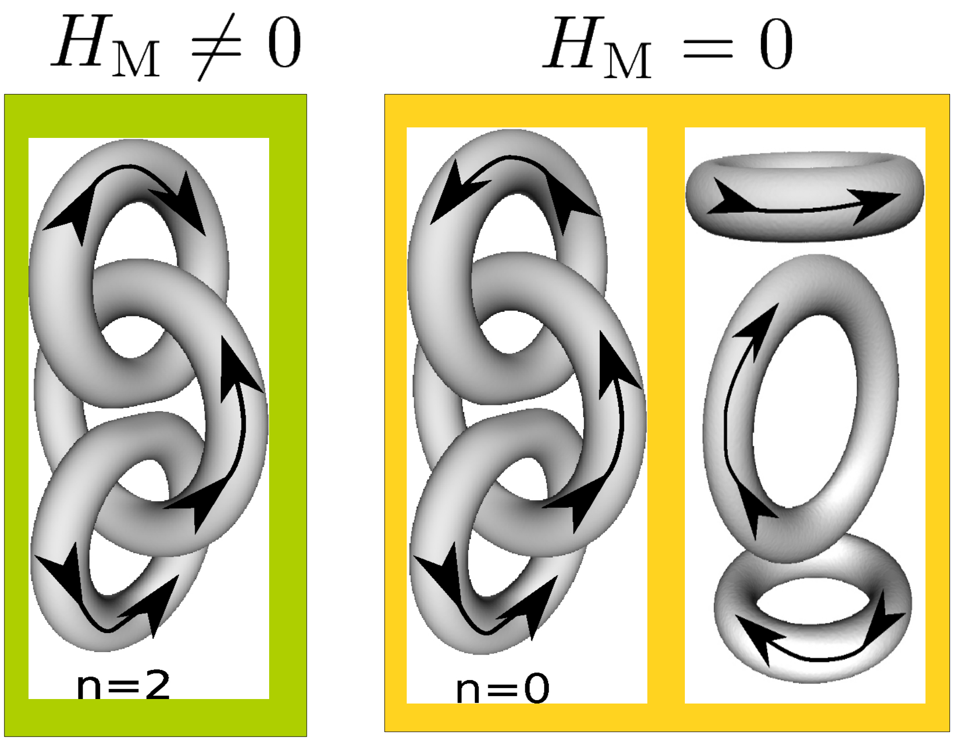
Realizability condition:

$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

→ Magnetic energy is bound from below by magnetic helicity.

Interlocked Flux Rings

actual linking vs. magnetic helicity



- initial condition: flux tubes
- isothermal compressible gas
- viscous medium
- periodic boundaries

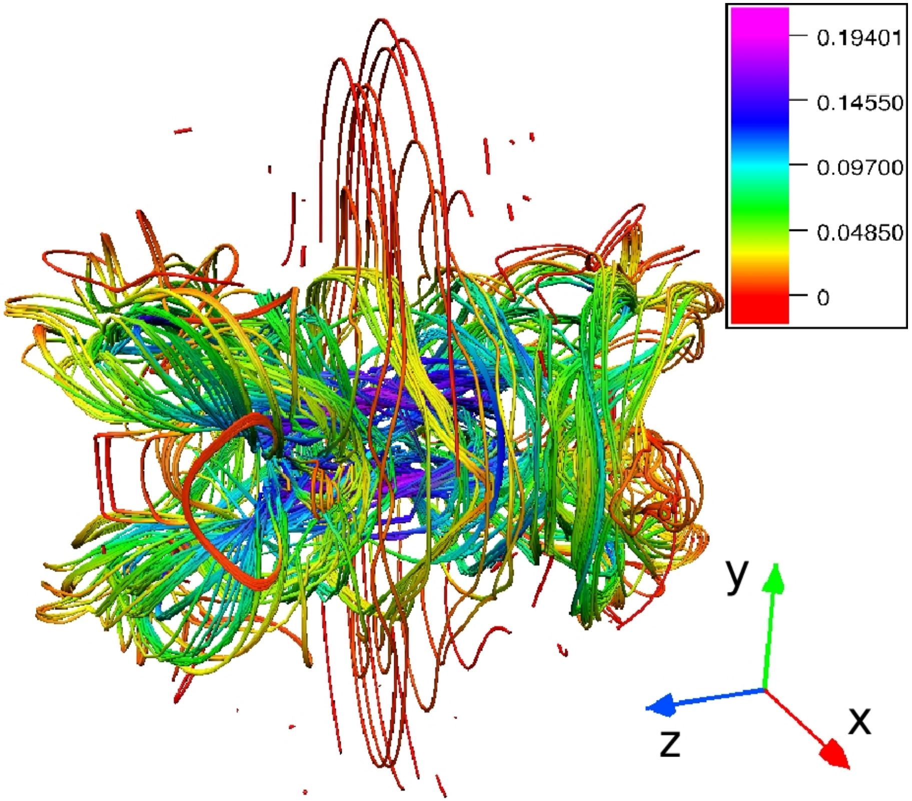
(Del Sordo et al. 2010)

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{U} \times \boldsymbol{B} + \eta \nabla^2 \boldsymbol{A} \qquad \frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \boldsymbol{U}$$

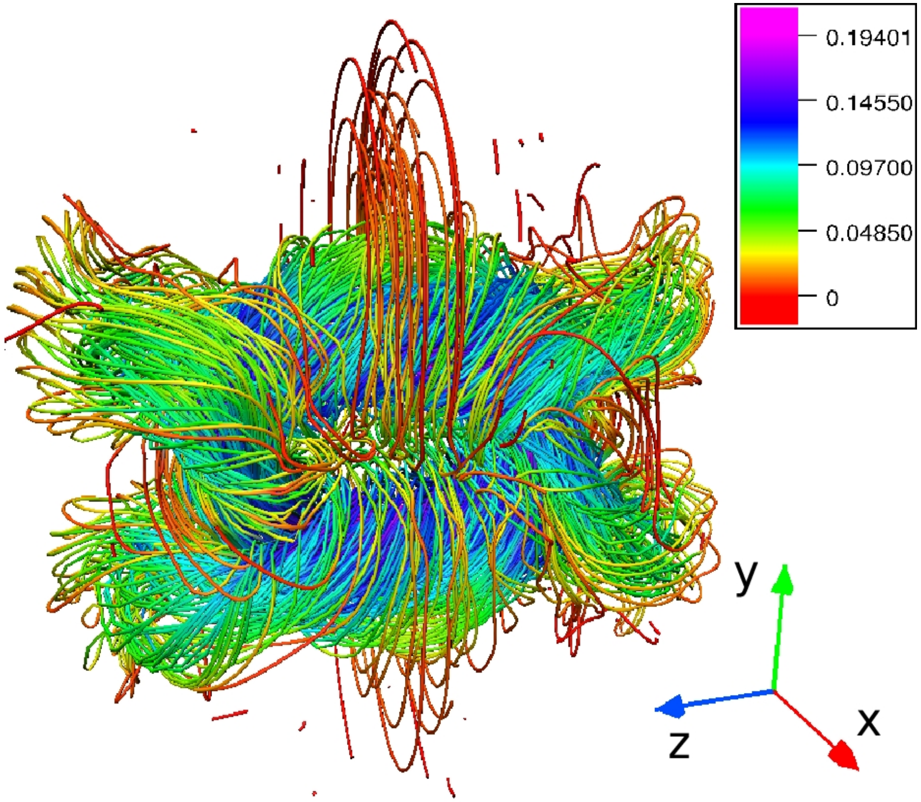
$$\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} = -c_{\rm S}^2 \boldsymbol{\nabla} \ln \rho + \boldsymbol{J} \times \boldsymbol{B}/\rho + \boldsymbol{F}_{\rm visc}$$

Interlocked Flux Rings

$\tau = 4$

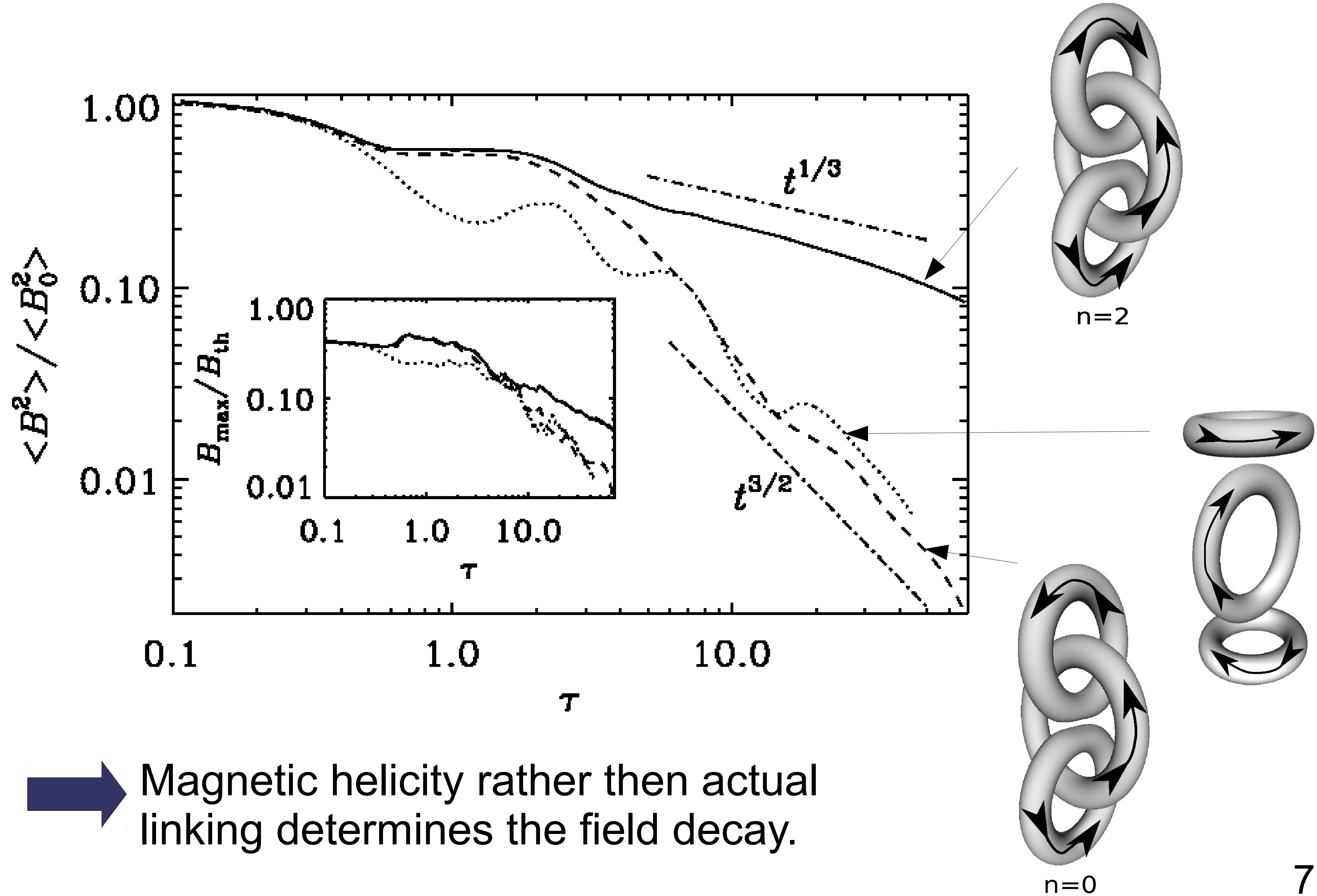


$H_{\mathrm{M}} = 0$



$H_{\mathrm{M}} \neq 0$

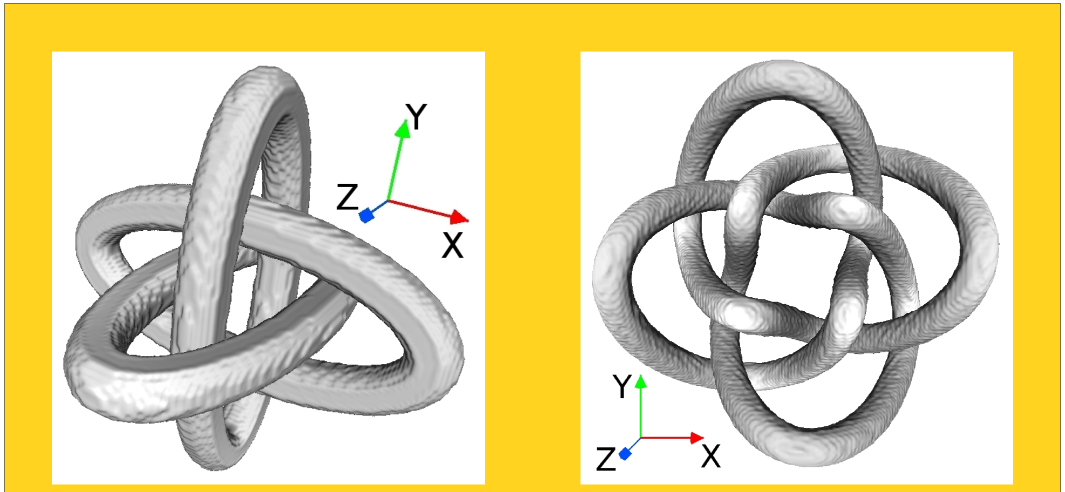
Interlocked Flux Rings



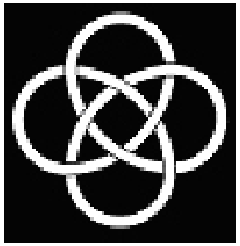
Magnetic helicity rather then actual linking determines the field decay.

IUCAA Knot and Borromean Rings

- Is magnetic helicity sufficient?
- Higher order invariants?



Borromean rings IUCAA knot

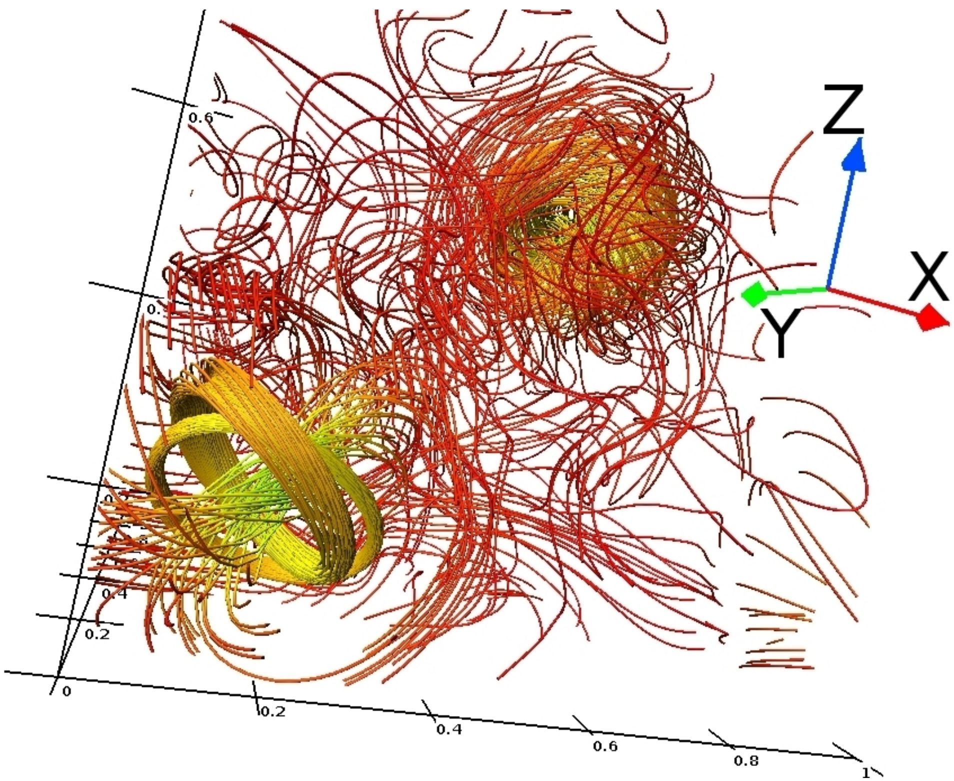


$$H_{\mathrm{M}} = 0$$

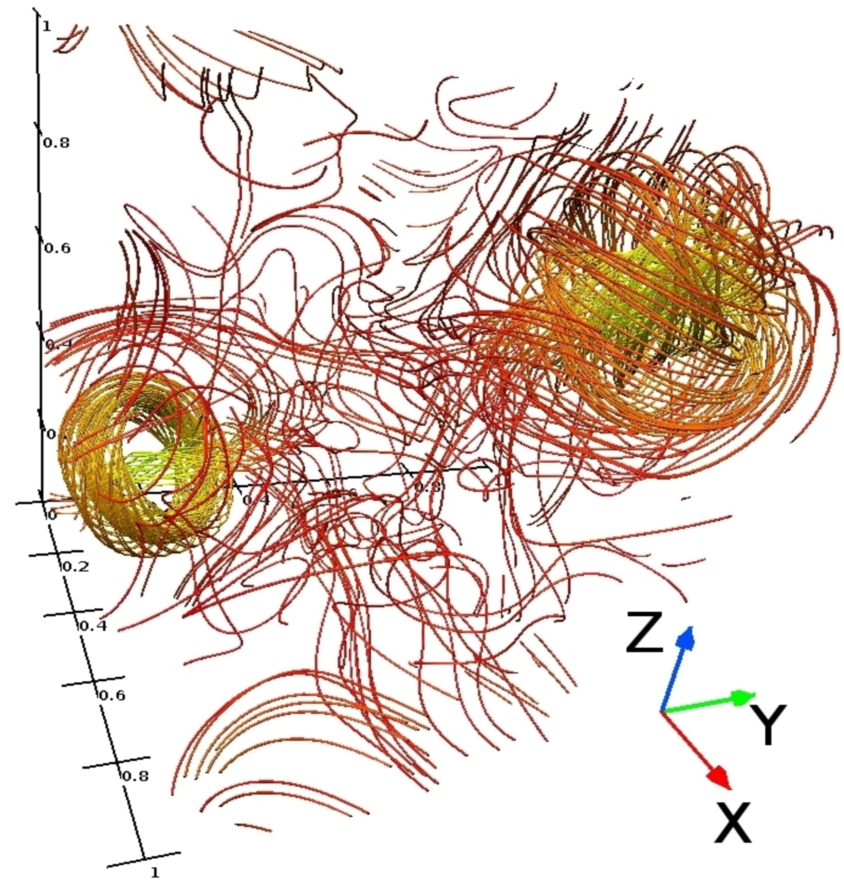
(Candelaresi and Brandenburg 2011)

IUCAA = The Inter-University Centre for Astronomy and Astrophysics, Pune, India

Reconnection Characteristics



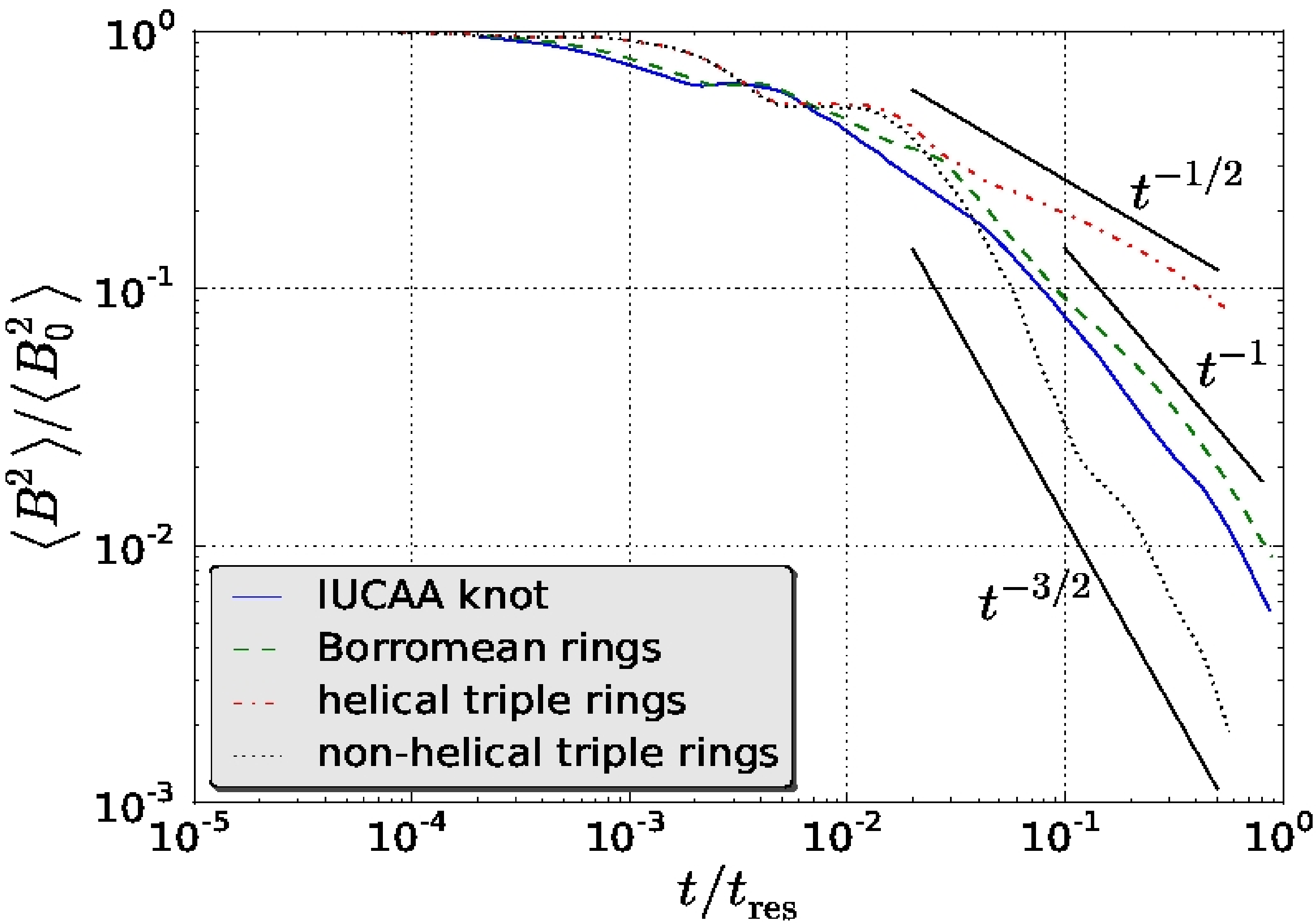
$t = 70$



$t = 78$

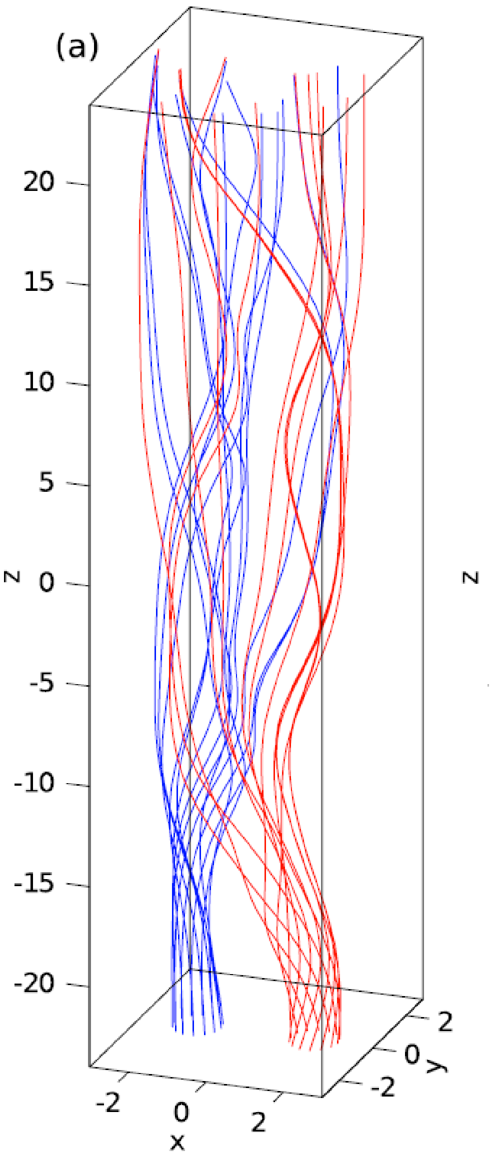
3 rings → Twisted ring + interlocked rings → 2 twisted rings

Magnetic Energy Decay



Higher order invariants?

Fixed Point Index



(Yeates et al. 2011)

Trace magnetic field lines from z_0 to z.

mapping: $(x, y) \rightarrow \mathbf{F}_z(x, y)$

fixed points: $\mathbf{F}_1(x, y) = (x, y)$

Color coding:

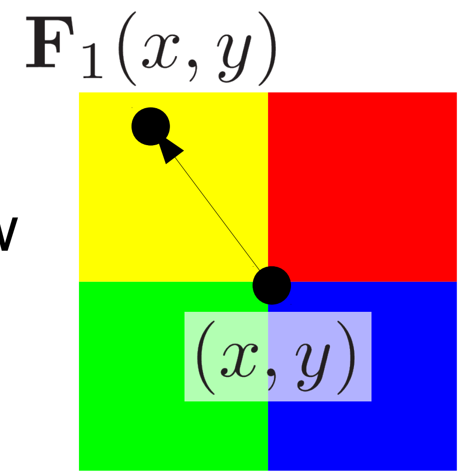
Compare (x, y) with $\mathbf{F}_1(x, y)$:

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y > y$ → red

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y > y$ → yellow

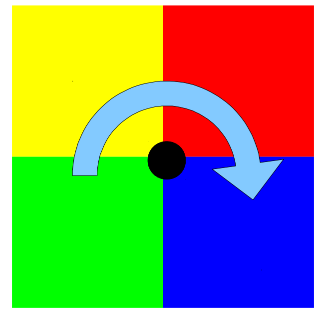
$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y < y$ → green

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y < y$ → blue

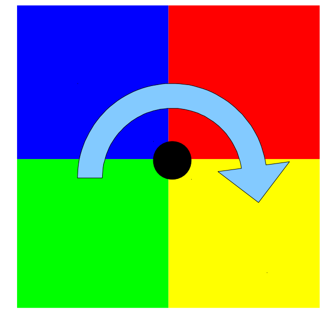


Fixed Point Index

Sign t_i of fixed point i :

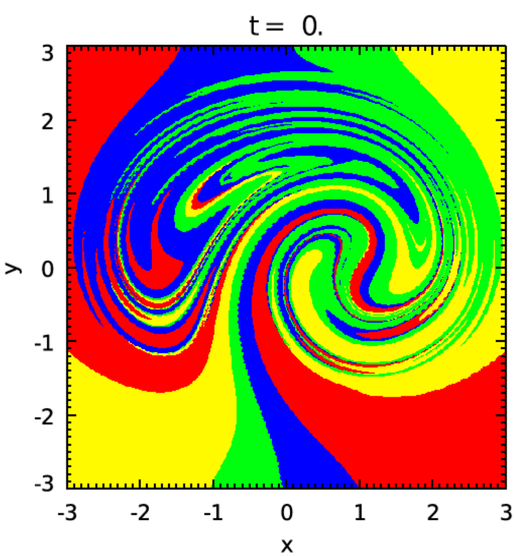


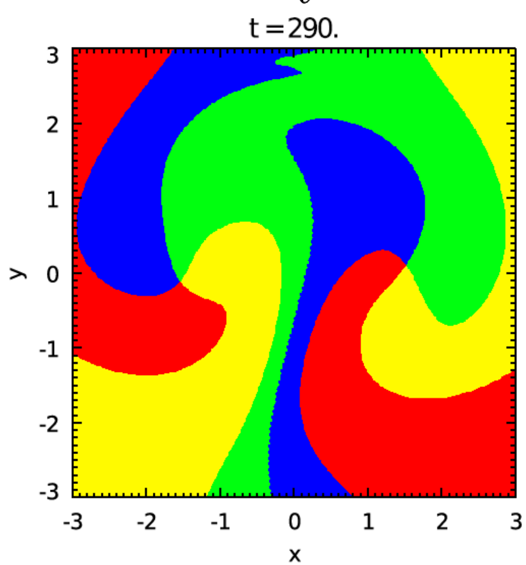
$t_i = +1$

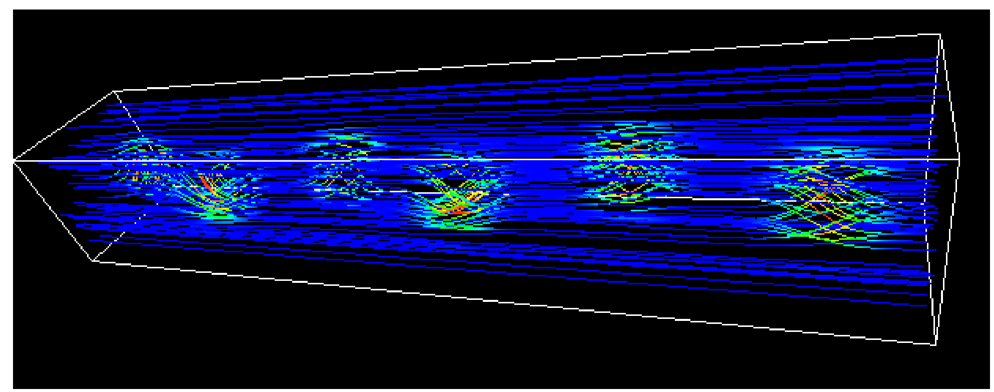


$t_i = -1$

Fixed point index: $T = \sum_i t_i$ conserved for $\lim \eta \to 0$







Taylor state is not reached
$\rightarrow$ T is additional constraint

Summary and Outlook

- Braiding increases stability through the *realizability condition*.
- Turbulent magnetic field decay is restricted by magnetic helicity.
- Fixed point index as additional constraint in relaxation.

- Use fixed point index for knots and links (Yeates A.).

References

Canfield et al. 1999

Canfield, R. C., Hudson, H. S., and McKenzie, D. E.
Sigmoidal morphology and eruptive solar activity.
Geophys. Res. Lett., 26:627, 1999

Del Sordo et al. 2010

Fabio Del Sordo, Simon Candelaresi, and Axel Brandenburg.
Magnetic-field decay of three interlocked flux rings with zero linking number.
Phys. Rev. E, 81:036401, Mar 2010.

Candelaresi and Brandenburg 2011

Simon Candelaresi, and Axel Brandenburg.
Decay of helical and non-helical magnetic knots.
Phys. Rev. E, 84:016406, 2011

Yeates et al. 2011

Yeates, A. R., Hornig, G. and Wilmot-Smith, A. L.
Topological Constraints on Magnetic Relaxation.
Phys. Rev. Lett. 105, 085002, 2010

Equilibrium States

Ideal MHD: $\eta = 0$

Induction equation: $\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B})$

Task: Find the state with minimal energy.
Constraint: magnetic helicity conservation

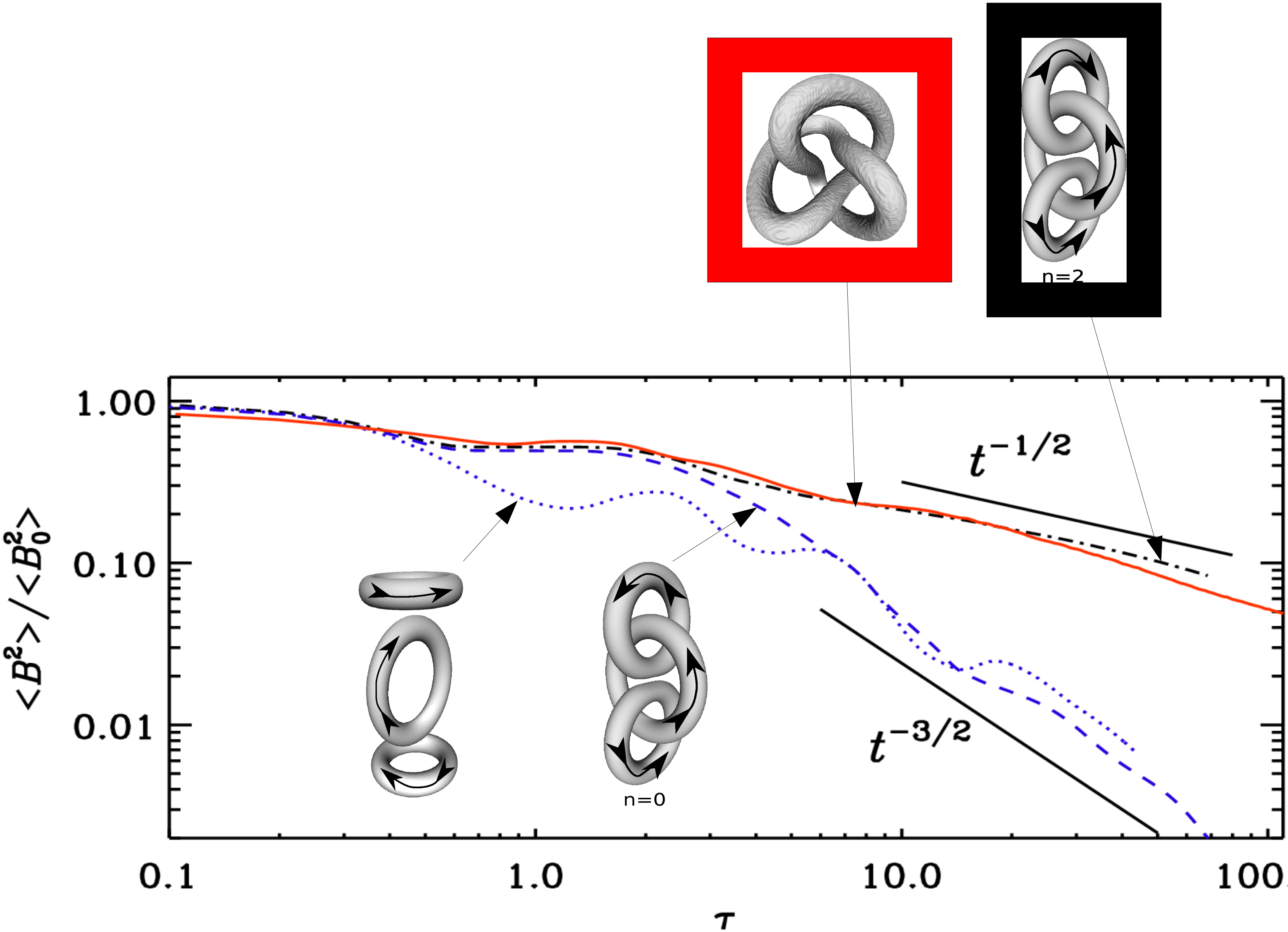
	constraint	equilibrium
Woltjer (1958):	$\frac{\partial}{\partial t} \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha \boldsymbol{B}$
Taylor (1974):	$\frac{\partial}{\partial t} \int_{\tilde{V}} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha(a,b) \boldsymbol{B}$

$\alpha(a,b)$: constant along field line

V total volume

$\tilde{V}$ volume along magnetic field line

Magnetic energy decay



Simulations

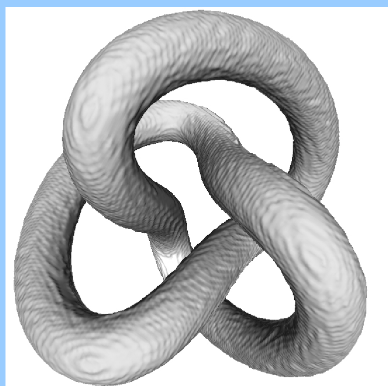
- 256^3 mesh point
- Isothermal compressible gas
- Viscous medium
- Periodic boundaries

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \eta \nabla^2 \mathbf{A}$$

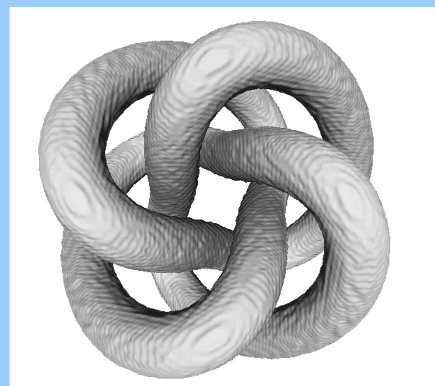
$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \boldsymbol{\nabla} \ln \rho + \mathbf{J} \times \mathbf{B}/\rho + \mathbf{F}_{\mathrm{visc}}$$

$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{U}$$

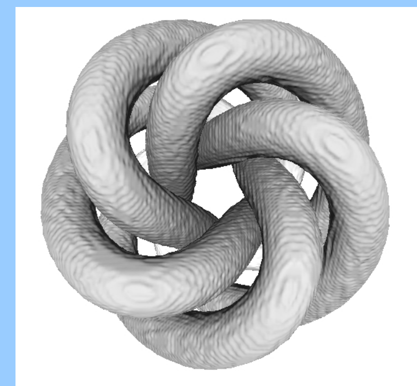
N-foil Knots



3-foil

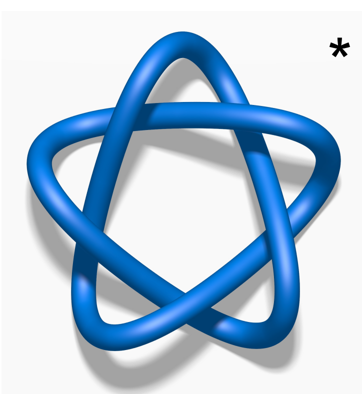


4-foil



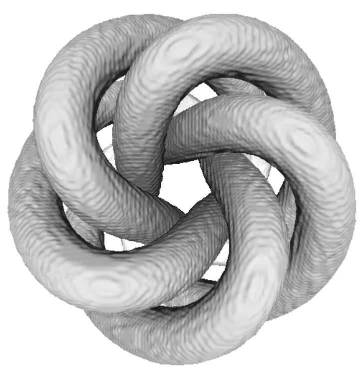
5-foil

6-foil 7-foil



cinquefoil knot

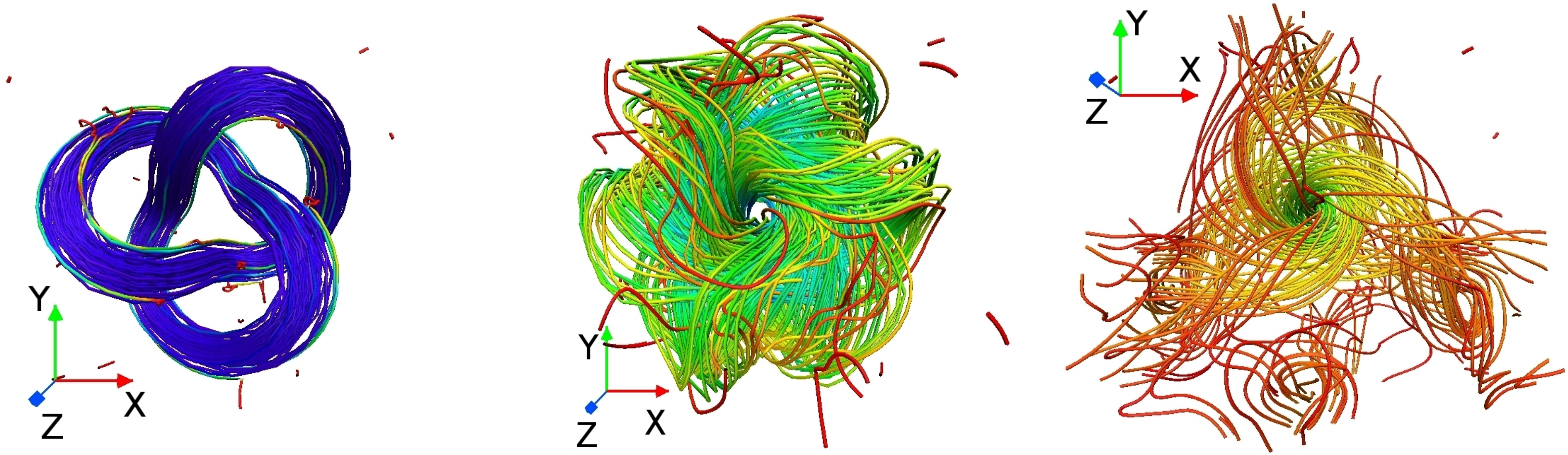
$\neq$



$$x(s) = \begin{pmatrix} (C + \sin sn_{\mathrm{f}}) \sin[s(n_{\mathrm{f}} - 1)] \\ (C + \sin sn_{\mathrm{f}}) \cos[s(n_{\mathrm{f}} - 1)] \\ D \cos sn_{\mathrm{f}} \end{pmatrix}$$

* from Wikipedia, author: Jim.belk

N-foil Knots

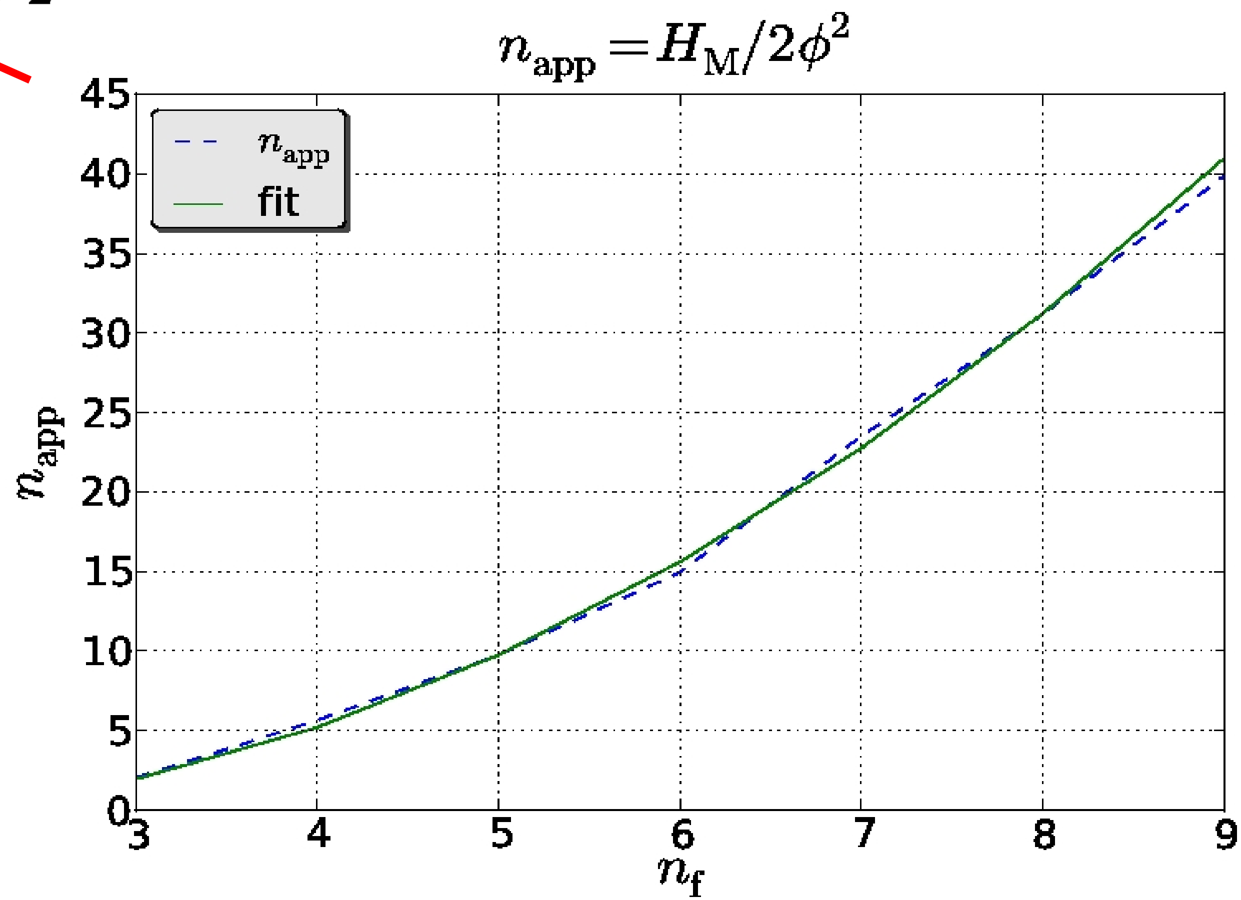


$t = 0$ $t = 6$ $t = 39$

- Magnetic helicity is approximately conserved.
- Self-linking is transformed into twisting after reconnection.

N-foil Knots

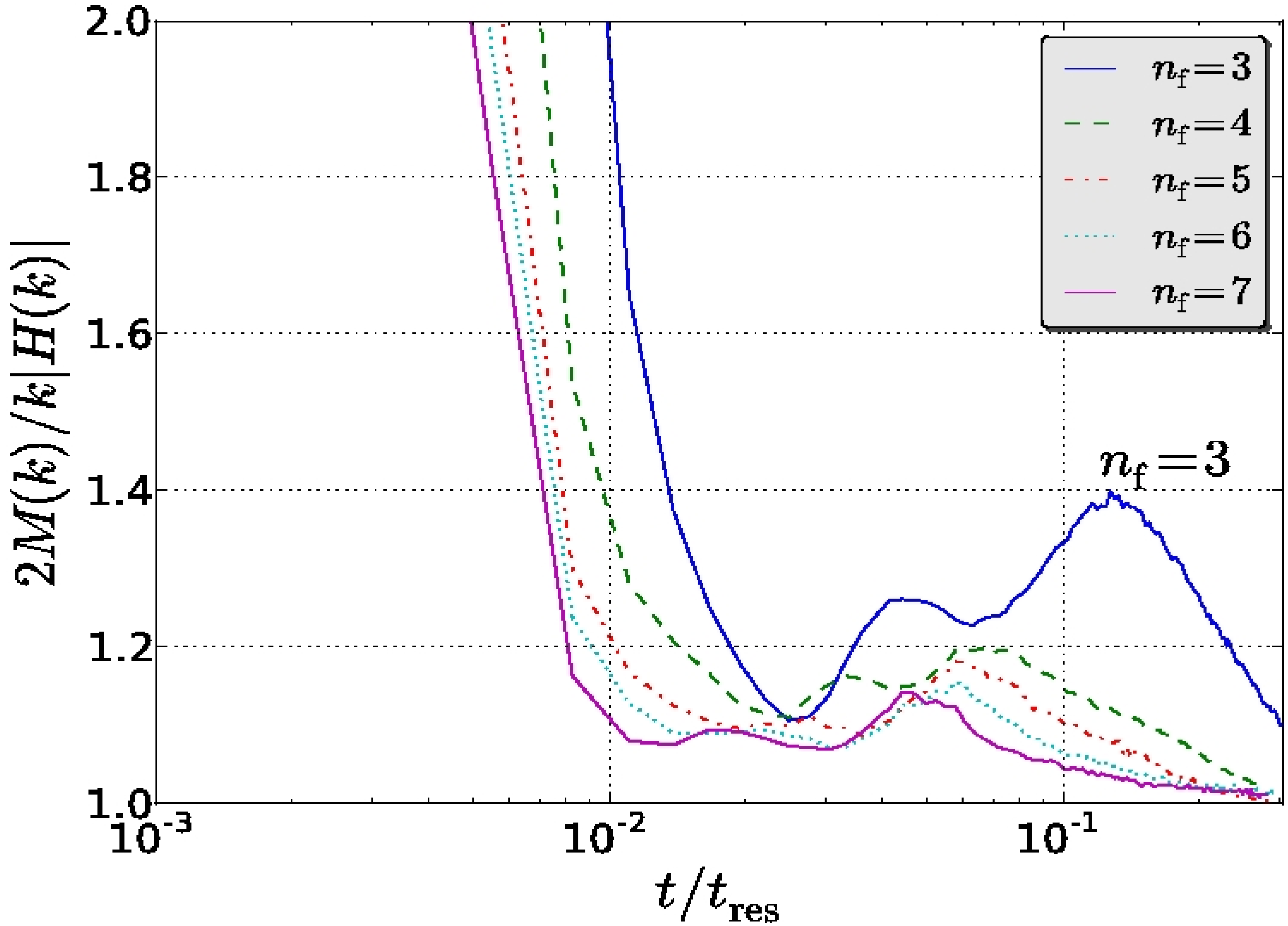
~~$H_{\mathrm{M}} = 2n\phi_1\phi_2$~~



$$H_{\mathrm{M}} = (n_{\mathrm{f}} - 2)n_{\mathrm{f}}\phi^2/2$$

N-foil Knots

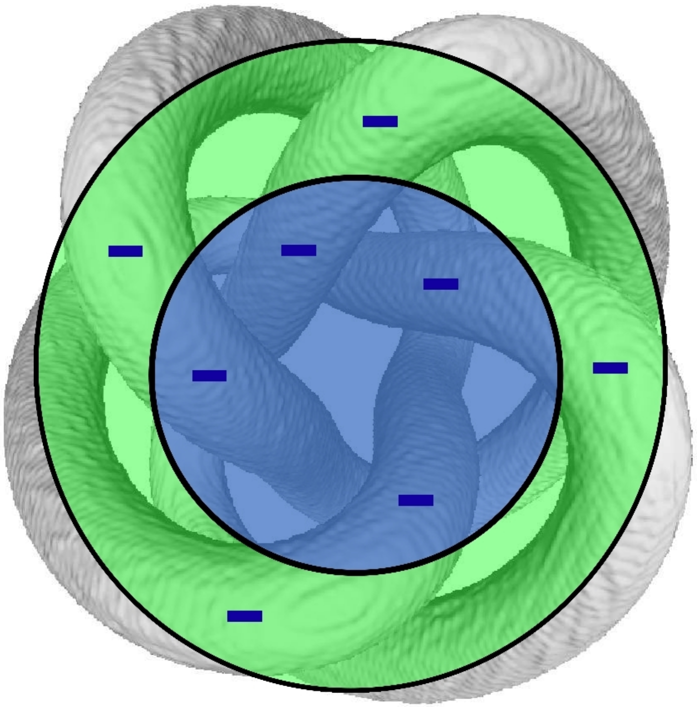
$$2M(k)/(|H(k)|k)$$



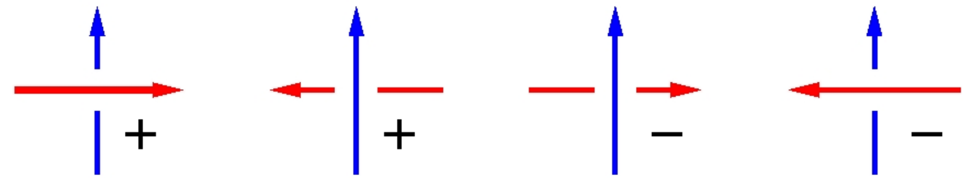
$k = 2$

Realizability condition more important for high n_{f}.

Linking Number



Sign of the crossings
for the 4-foil knot



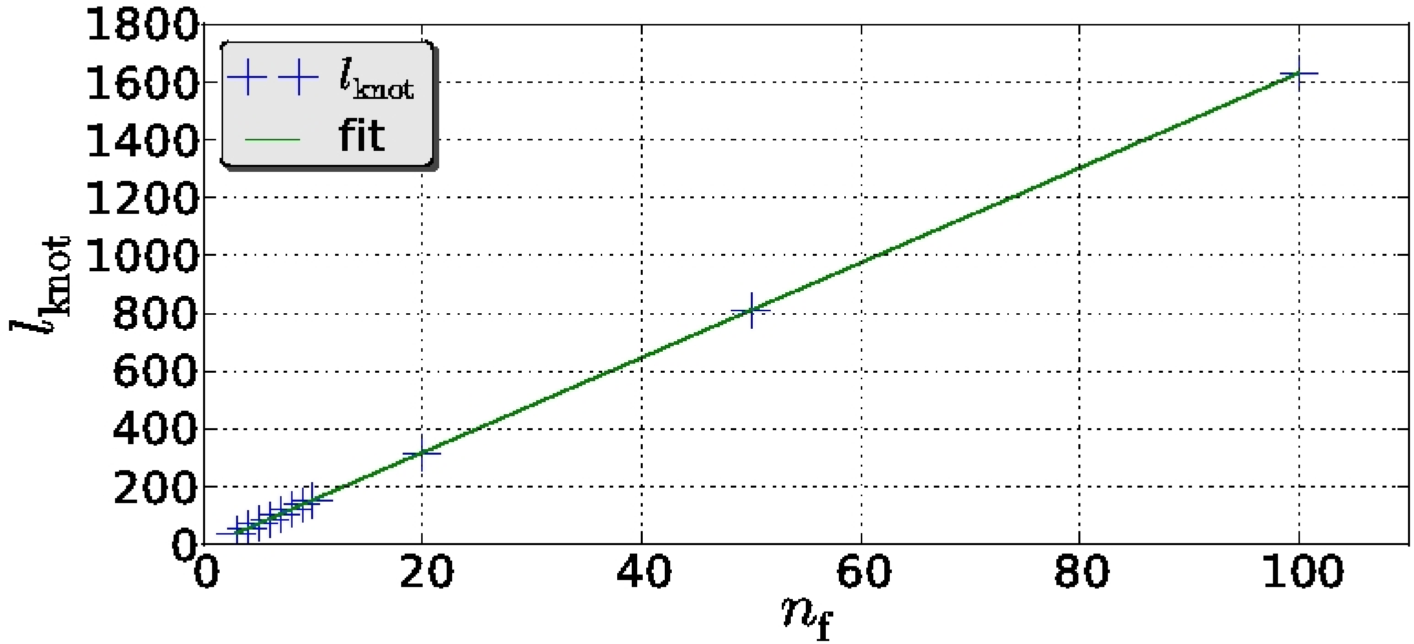
$$n_{\text{linking}} = (n_+ - n_-)/2$$

Number of crossings increases like n_f^2

$$H_M \propto n_{\text{linking}}$$

$$H_M \propto n_f^2$$

Helicity vs. Energy



$$E_{\mathrm{M}} \propto l_{\mathrm{knot}} \propto n_{\mathrm{f}}$$

$$H_{\mathrm{M}} \propto n_{\mathrm{f}}^2$$

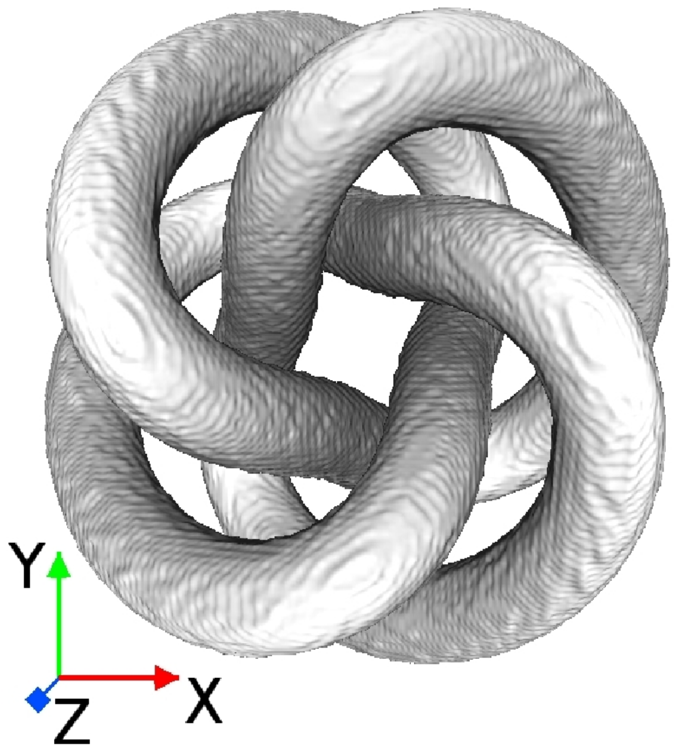
→ Knot is more strongly packed with increasing n_{f}.

↓

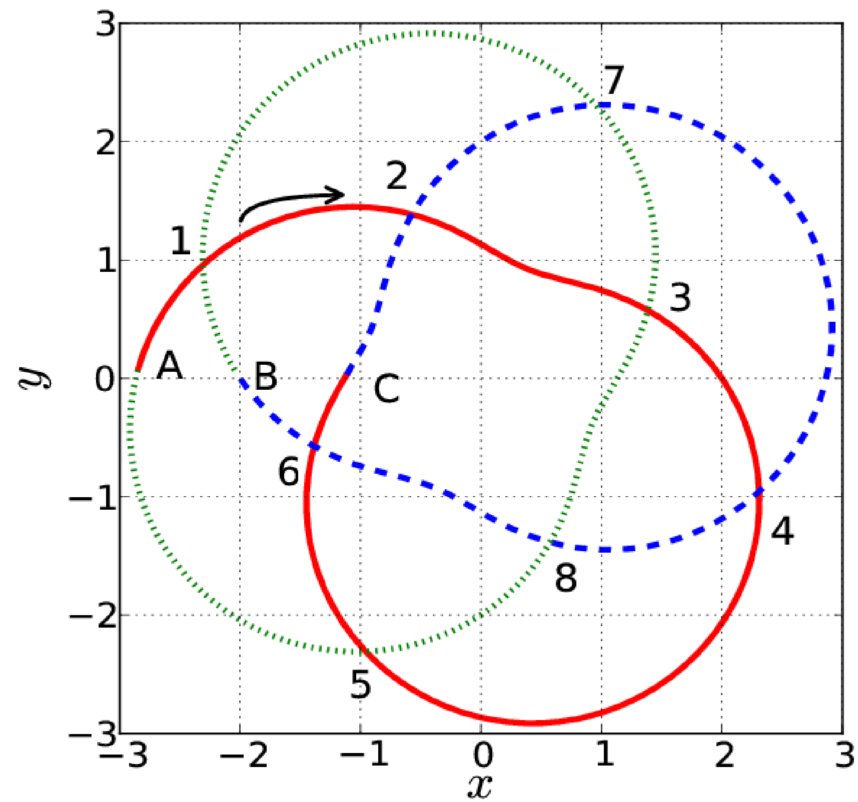
Magnetic energy is closer to its lower limit for high n_{f}.

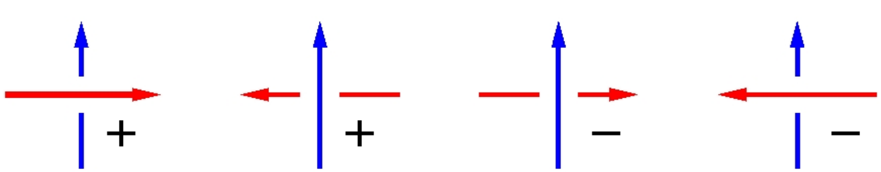
Braid Representation

need $B_z > 0$ → braid representation of knots and links

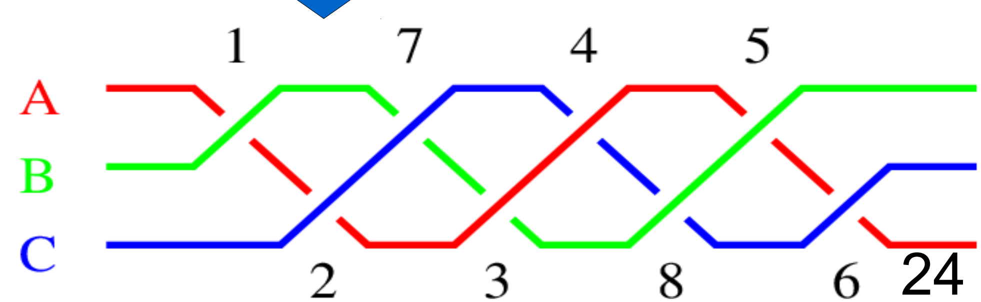


4-foil knot





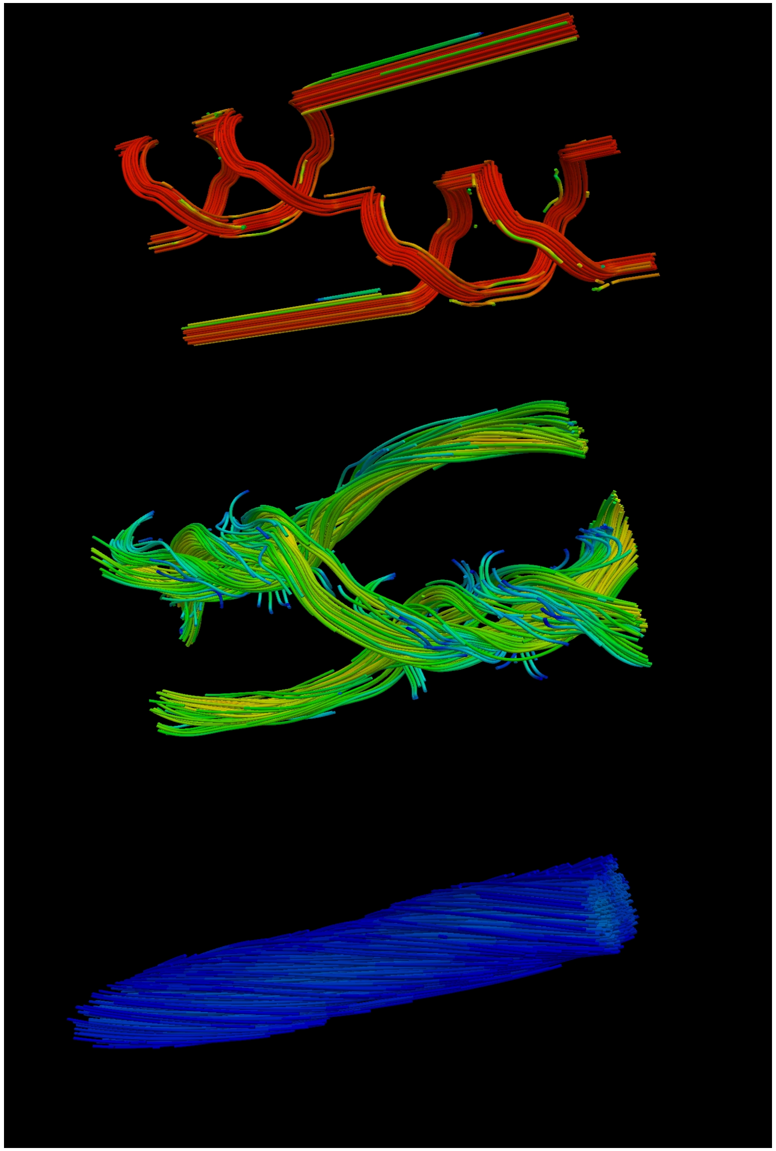
$$n_{\text{linking}} = (n_+ - n_-)/2$$



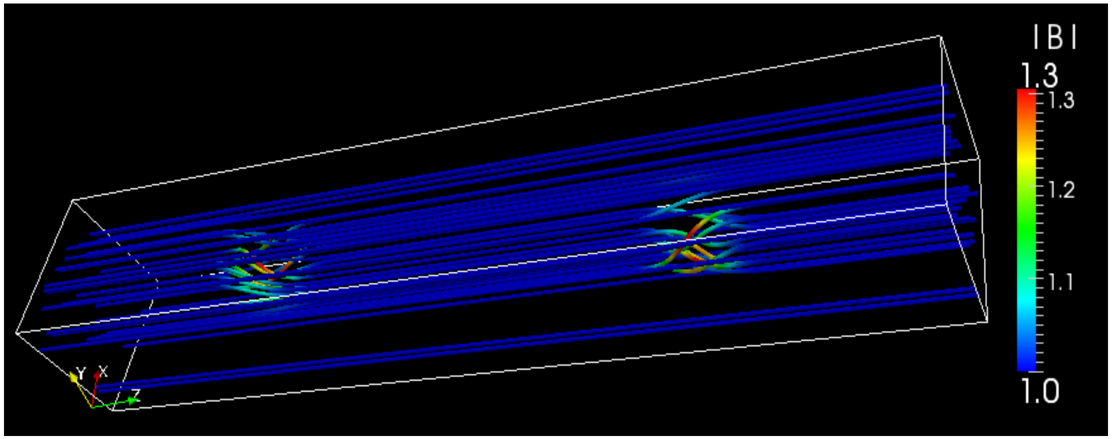
Magnetic Braid Configurations

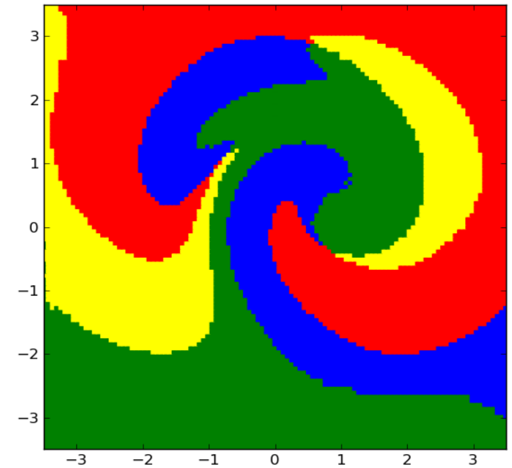
AAA (trefoil knot)

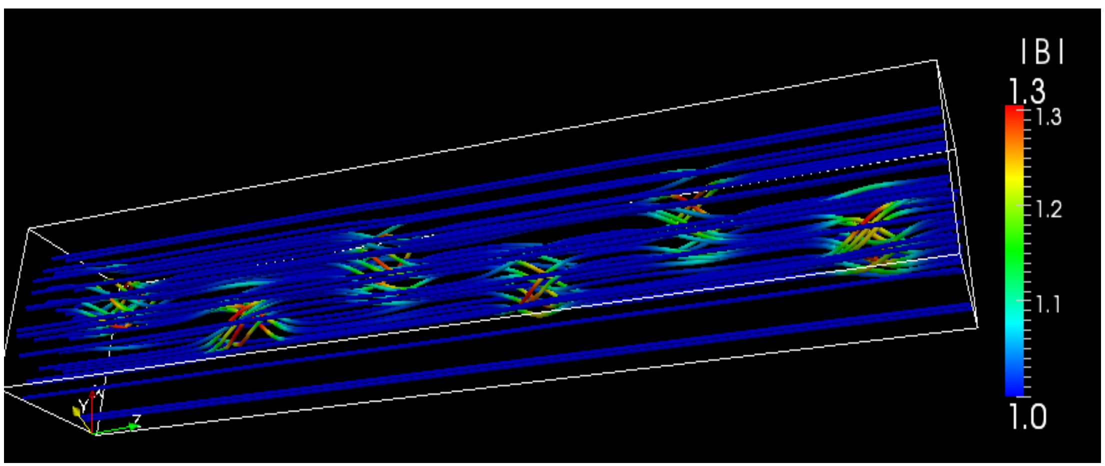
AABB (Borromean rings)

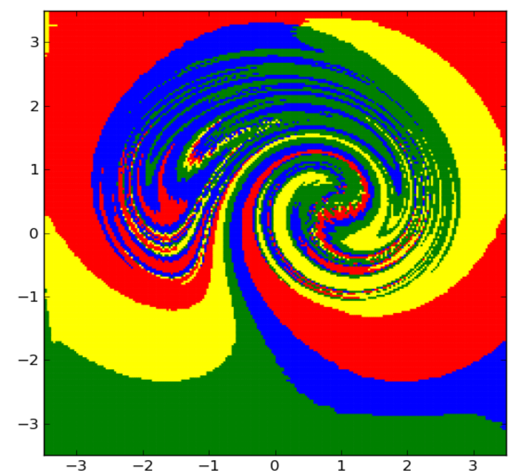


Field Line Tracing









Generalized flux function:

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

Reconnection rate:

$$\sum_i \frac{\mathrm{d}\mathcal{A}(\mathbf{x}_i)}{\mathrm{d}t}$$

Magnetic Reconnection Rate

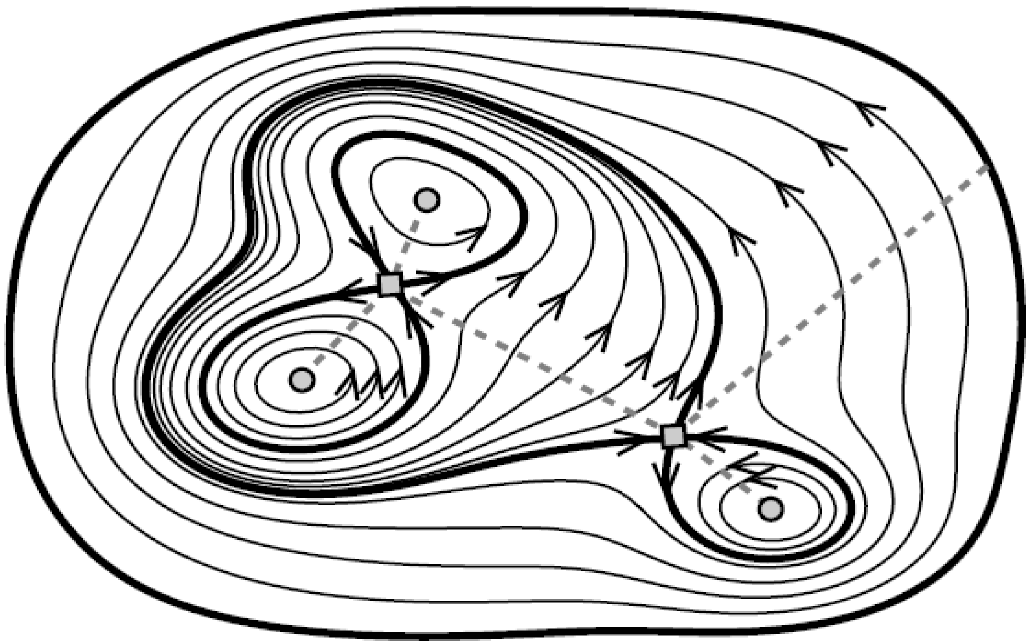
Classic: look for local maxima of $\int \boldsymbol{E} \cdot \boldsymbol{B}$

Partition fluxes 2D:
(Yeates, Hornig 2011b)

$$\boldsymbol{B} = \boldsymbol{\nabla} \times (A\boldsymbol{e}_z)$$

Reconnection rate = magnetic flux through boundaries (spearatrices):

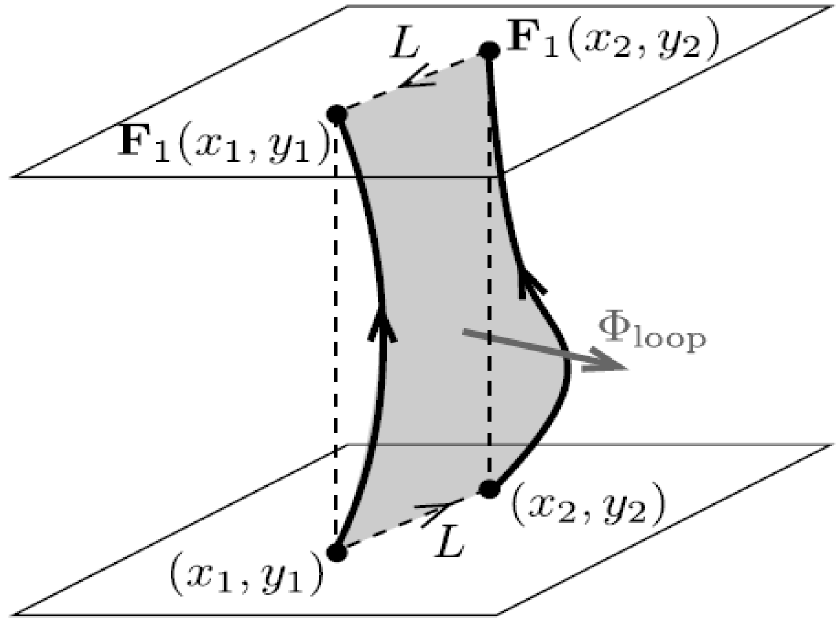
$$\Delta\phi = \sum_i \left| \frac{\mathrm{d}A(\boldsymbol{h}_i)}{\mathrm{d}t} \right|$$



2D Magnetic field.
Thick lines: separatrices.
(Yeates, Hornig 2011b)

Magnetic Reconnection Rate

Partition reconnection rate 3D:
Yeates, Hornig 2011b



Generalized flux function (curly A):

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \boldsymbol{A}\cdot\boldsymbol{B}/B_z \,\mathrm{d}z$$

$$\phi = \int_S \boldsymbol{\nabla}\times\boldsymbol{A}\cdot\ \mathrm{d}\boldsymbol{s} = \int_C \boldsymbol{A}\cdot\ \mathrm{d}\boldsymbol{l}$$

$$\frac{\partial\mathcal{A}}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\mathcal{A} = 0$$

invariant in ideal MHD

Fixed points: $\mathbf{F}_1(x_i, y_i) = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$

Reconnection rate:

$$\Delta\phi = \sum_i \left|\frac{\mathrm{d}\mathcal{A}(\boldsymbol{h}_i)}{\mathrm{d}t}\right|$$