How much helicity is needed to drive large-scale dynamos?

(submitted to *Phys. Rev. E*)

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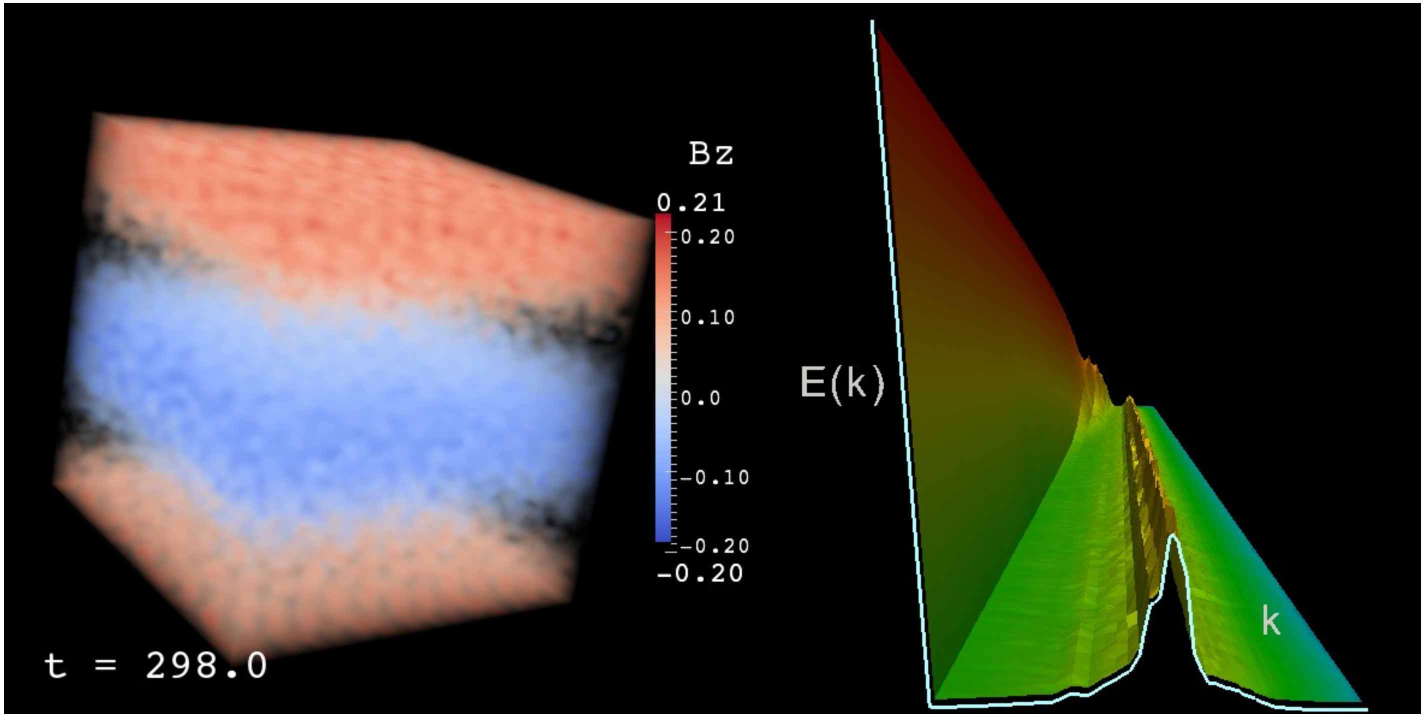


Closed alpha^2 Dynamo

Momentum equation:

forcing function

$$\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{U} = -c_\mathrm{s}^2 \boldsymbol{\nabla} \ln \rho + \frac{1}{\rho}\boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{F}_\mathrm{visc} + \boldsymbol{f}$$

Helical forcing $\boldsymbol{f}$ on scale k_f

↓

Helical motions $\langle \boldsymbol{\omega} \cdot \boldsymbol{u} \rangle \approx \epsilon_\mathrm{f} k_\mathrm{f} \langle \boldsymbol{u}^2 \rangle$

↓

Helical magnetic field $\langle \overline{\boldsymbol{J}} \cdot \overline{\boldsymbol{B}} \rangle = \epsilon_\mathrm{m} k_\mathrm{m} \langle \overline{\boldsymbol{B}}^2 \rangle$

$$= (\epsilon_\mathrm{m} k_\mathrm{m})^2 \langle \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{B}} \rangle$$

(Frisch et. al. 1975, Seehafer 1996)

$\epsilon_\mathrm{f}, \epsilon_\mathrm{m}$ = normalized helicities

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$$

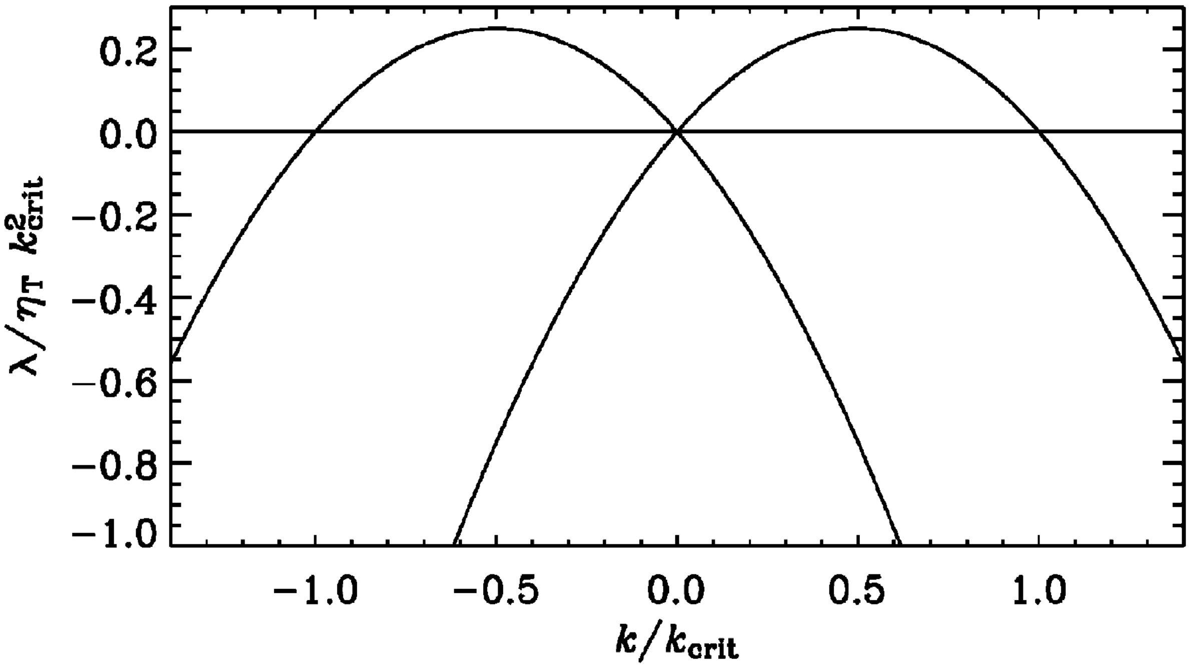
Predictions from the General Theory

Kinematic phase: $\lambda = \pm\alpha k - \eta_\mathrm{T} k^2 = (\pm C_\alpha - 1)\eta_\mathrm{T} k^2$

(Moffatt 1978)

$$\alpha = -(\tau/3)\langle \boldsymbol{\omega}\cdot\boldsymbol{u}\rangle \qquad \eta_\mathrm{T} = \eta + \eta_\mathrm{t}$$

$$C_\alpha = \frac{\alpha}{\eta_\mathrm{T} k} \qquad \eta_\mathrm{t} = \frac{\tau}{3}\langle \boldsymbol{u}^2\rangle \qquad \tau = (u_\mathrm{rms} k_\mathrm{f})^{-1}$$



(Brandenburg, Subramanian 2005)

Predictions from the General Theory

mean-field interpretation

Mean-field decomposition: $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$

Induced small-scale helical motions: $\alpha_{\mathrm{K}} = -\frac{\langle \boldsymbol{\omega} \cdot \boldsymbol{u} \rangle}{3 u_{\mathrm{rms}} k_{\mathrm{f}}}$

(Krause, Raedler 1980)

Magnetic helicity conservation: $\alpha = -\frac{\langle \boldsymbol{\omega} \cdot \boldsymbol{u} \rangle + \langle \boldsymbol{j} \cdot \boldsymbol{b} \rangle / \langle \rho \rangle}{3 u_{\mathrm{rms}} k_{\mathrm{f}}}$

(Pouquet et. al. 1976)

Triply periodic BC → Magnetic helicity is conserved.

→ $t_{\mathrm{sat}} = t_{\mathrm{res}} = (2 \eta \epsilon_{\mathrm{m}}^2 k_1^2)^{-1}$

resistive growth for large-scale field $\overline{\boldsymbol{B}}$ *(Brandenburg, Subramanian 2005)*

Predictions from the General Theory

mean-field interpretation

Saturation magnetic field strength:

$$\overline{B}^2_{\rm sat}/B^2_{\rm eq} = (C_\alpha/\epsilon_{\rm m} - 1)\iota$$ *(Blackman, Brandenburg 2002)*

$$\iota = \eta_{\rm T}/\eta_{\rm t} = (1 + 3/{\rm Re}_{\rm M})$$

$${\rm Re}_{\rm M} = \frac{u_{\rm rms}}{\eta k_{\rm f}} \qquad B_{\rm eq} = u_{\rm rms}(\mu_0\overline{\rho})^{1/2}$$

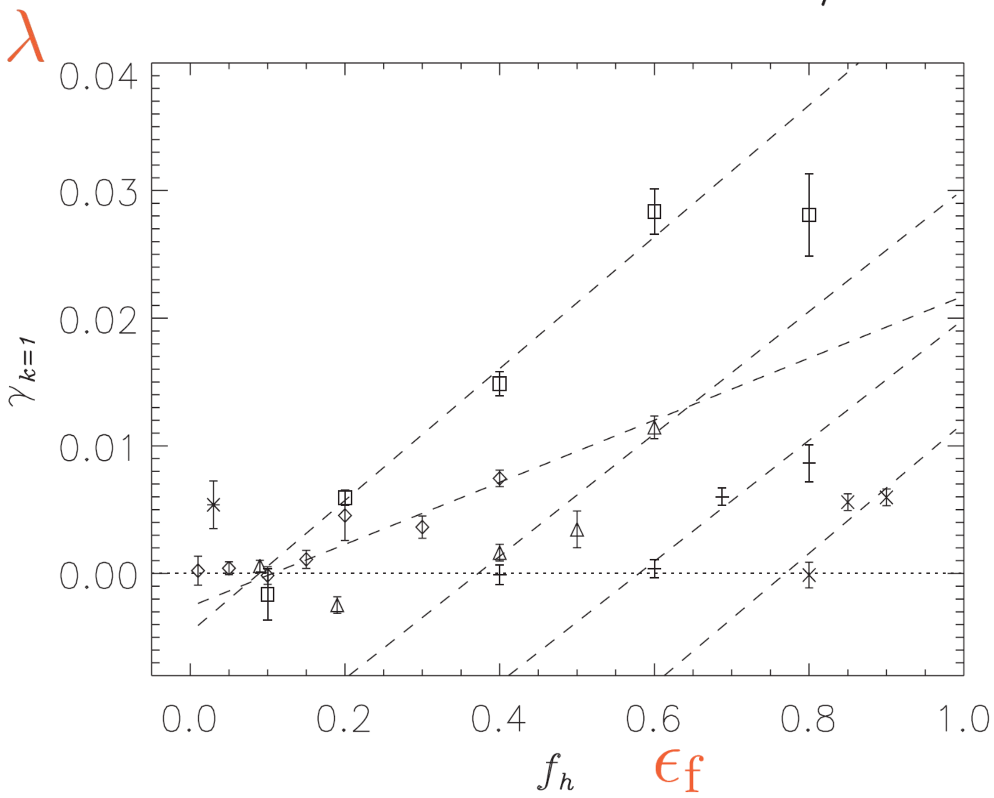
For the man magnetic field to grow: $|C_\alpha^{\rm crit}| = \epsilon_{\rm m}$

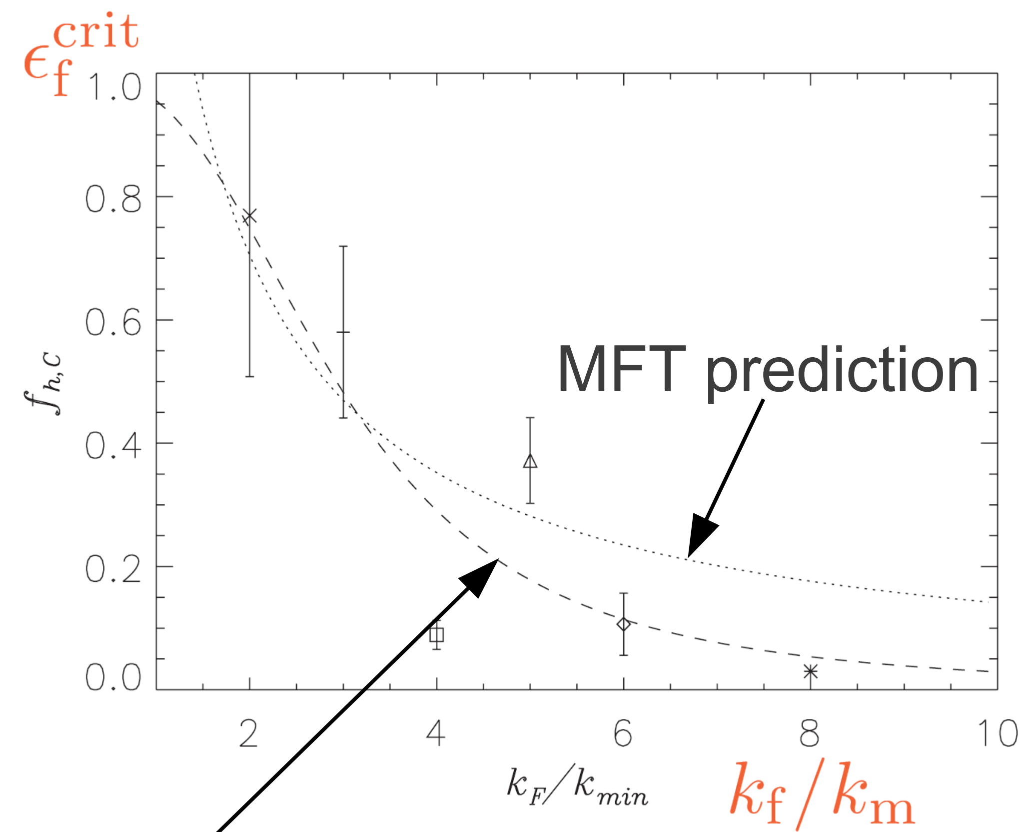
$$C_\alpha = -\frac{\langle \boldsymbol{\omega}\cdot\boldsymbol{u}\rangle}{\iota k_{\rm f} u^2_{\rm rms}} = -\frac{\epsilon_{\rm f} k_{\rm f}}{\iota k_{\rm m}} \quad \Rightarrow \quad \epsilon_{\rm f}^{\rm crit} = \iota\epsilon_{\rm m}\left(\frac{k_{\rm f}}{k_{\rm m}}\right)^{-1}$$

$$\epsilon_{\rm f} = \frac{\langle \boldsymbol{\omega}\cdot\boldsymbol{u}\rangle}{k_{\rm f} u_{\rm rms}} =$$ normalized kinetic helicity

What Pietarila Graham Finds

Parameters: $\epsilon_{\rm f}$ and $k_{\rm f}/k_{\rm m}$





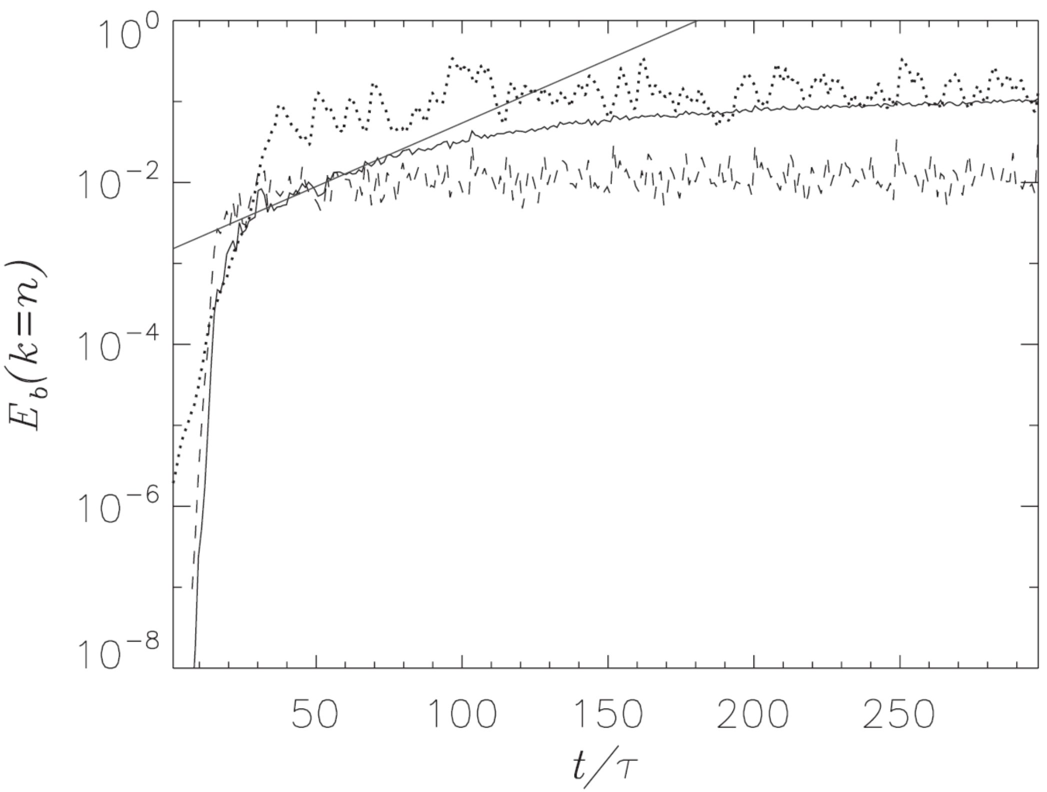
Fit formula: $f_{\rm h,C} = 1/\left(1 + C^2 (k_{\rm f}/k_{\rm m})^{2\xi+2}\right)$ $\quad \xi \approx 0.46$

(Pietarila Graham, et. al. 2012)

$$\left(\frac{k_{\rm f}}{k_{\rm m}}\right)^{-1} \neq \left(\frac{k_{\rm f}}{k_{\rm m}}\right)^{-3}$$

Possible Issues

- Growth rates after a fraction of the resistive time.
- Dynamo still contaminated with magnetic fields from the small-scale dynamo.
- Inaccurate fit for $f_{\mathrm{h,C}}$.



(Pietarila Graham, et. al. 2012)

Reproduction of the Predictions

Consider the resistive phase well after the kinematic phase.

$$\frac{\partial}{\partial t}\boldsymbol{A} = \boldsymbol{U} \times \boldsymbol{B} - \eta\mu_0\boldsymbol{J}$$

forcing function

$$\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{U} = -c_{\mathrm{s}}^2\boldsymbol{\nabla}\ln\rho + \frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B} + \boldsymbol{F}_{\mathrm{visc}} + \boldsymbol{f}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\ln\rho = -\boldsymbol{\nabla}\cdot\boldsymbol{U}$$

triply periodic BC → magnetic helicity is conserved

helical forcing $\boldsymbol{f}$ → helical motions $\langle\boldsymbol{\omega}\cdot\boldsymbol{u}\rangle \approx \epsilon_{\mathrm{f}}k_{\mathrm{f}}\langle\boldsymbol{u}\rangle$

→ helical magnetic field (Beltrami field) $\langle\overline{\boldsymbol{J}}\cdot\overline{\boldsymbol{B}}\rangle = k_{\mathrm{m}}\langle\overline{\boldsymbol{B}}^2\rangle$

Parameters: ϵ_{f} and $k_{\mathrm{f}}/k_{\mathrm{m}}$

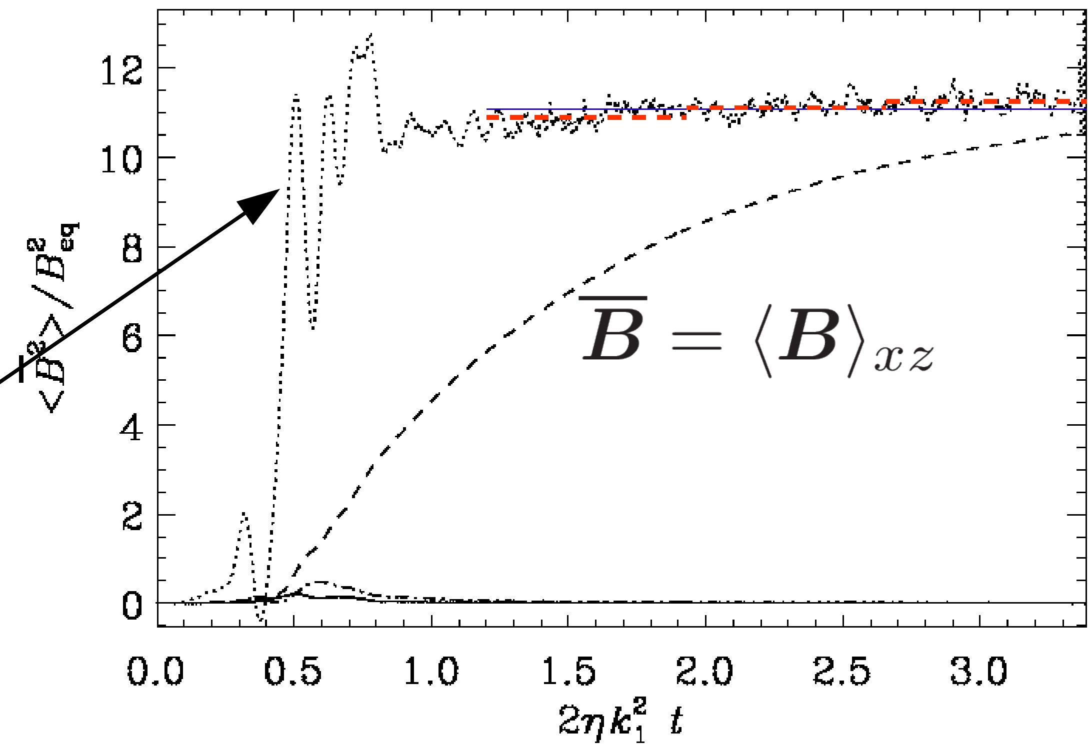
Saturation Magnetic Field

resistive growth:

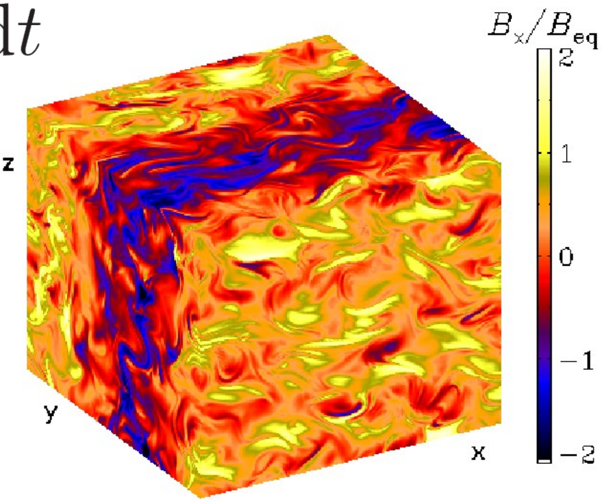
$$M(t) = M_0 - M_1 e^{-t/\tau}$$

$$\tau = (2\eta\epsilon_{\rm m}^2 k_{\rm m}^2)^{-1}$$

$$M_0 = M(t) + \tau \frac{\mathrm{d}}{\mathrm{d}t} M(t)$$



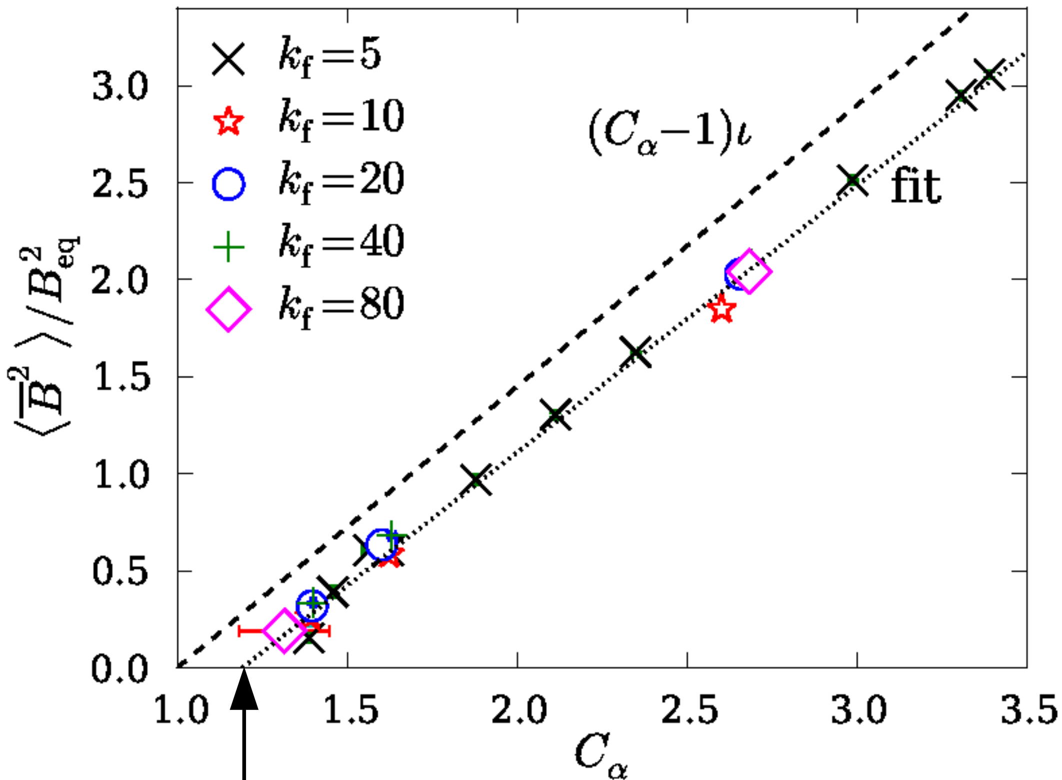
$$B_{\rm sat}^2 = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} M(t) + \tau \frac{\mathrm{d}}{\mathrm{d}t} M(t)\ \mathrm{d}t$$



(Candelaresi, Brandenburg 2012)

Saturation Magnetic Field

Prediction: $\overline{B}_{\rm sat}^2/B_{\rm eq}^2 = (C_\alpha - 1)\iota$ *(Blackman, Brandenburg 2002)*



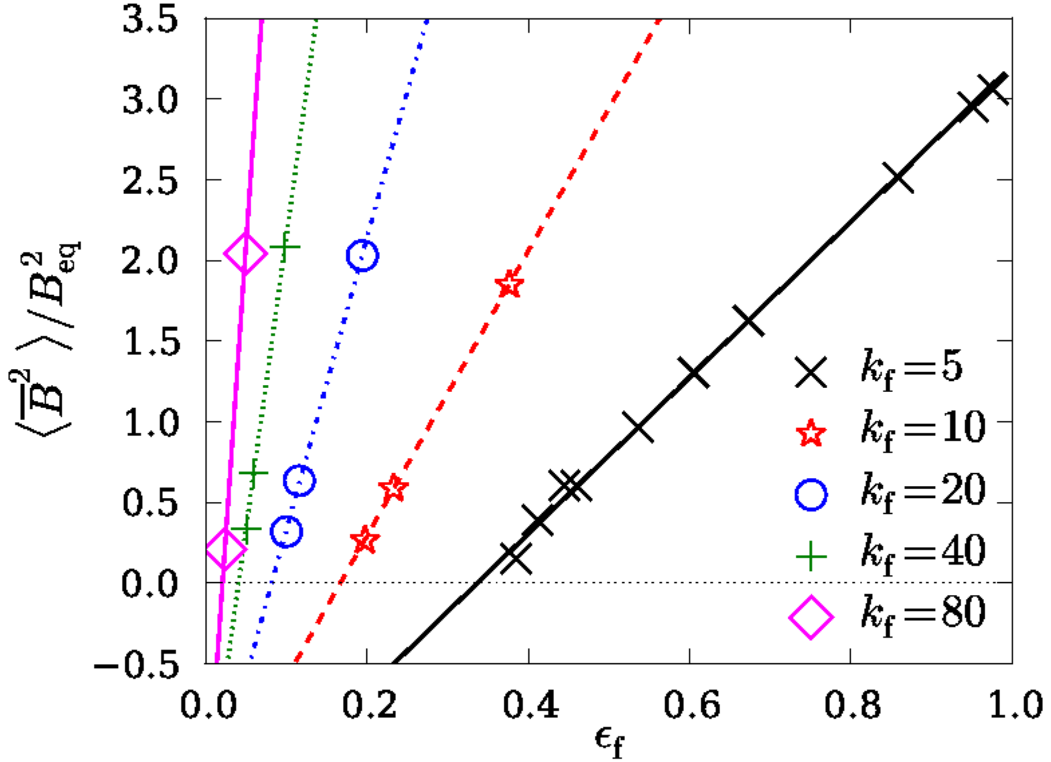
$C_\alpha^{\rm crit} \approx 1.2$

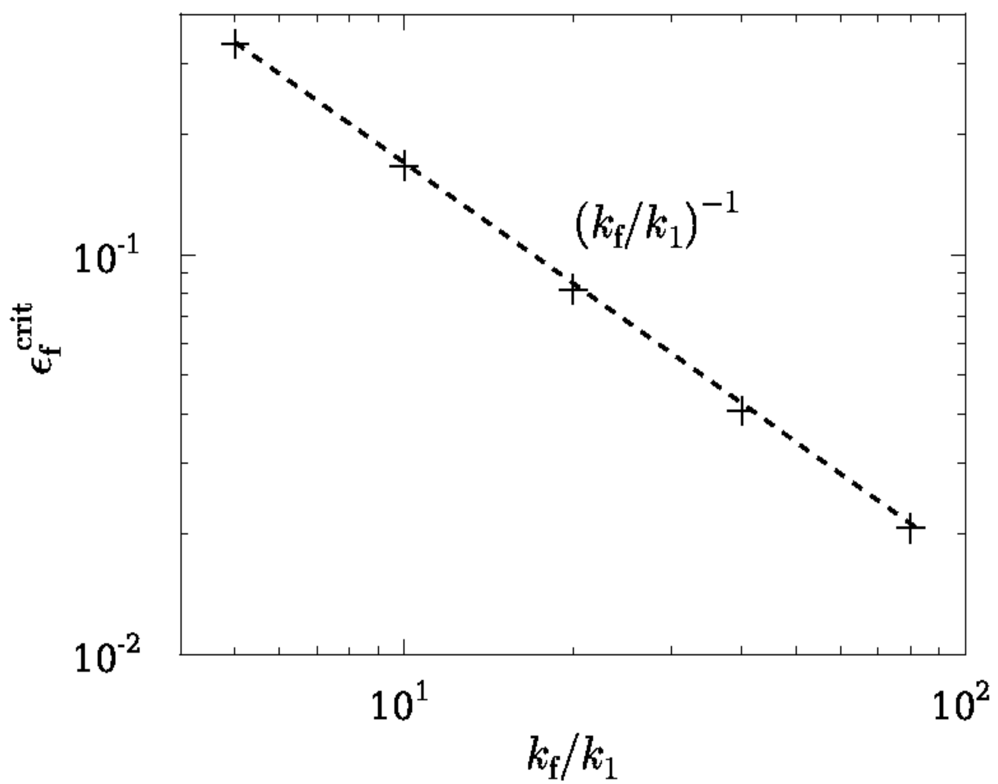
$\mathrm{Re_M} \approx 6$

$\mathrm{Pr_M} = \nu/\mu = 1$

(Candelaresi, Brandenburg 2012)

Saturation Magnetic Field





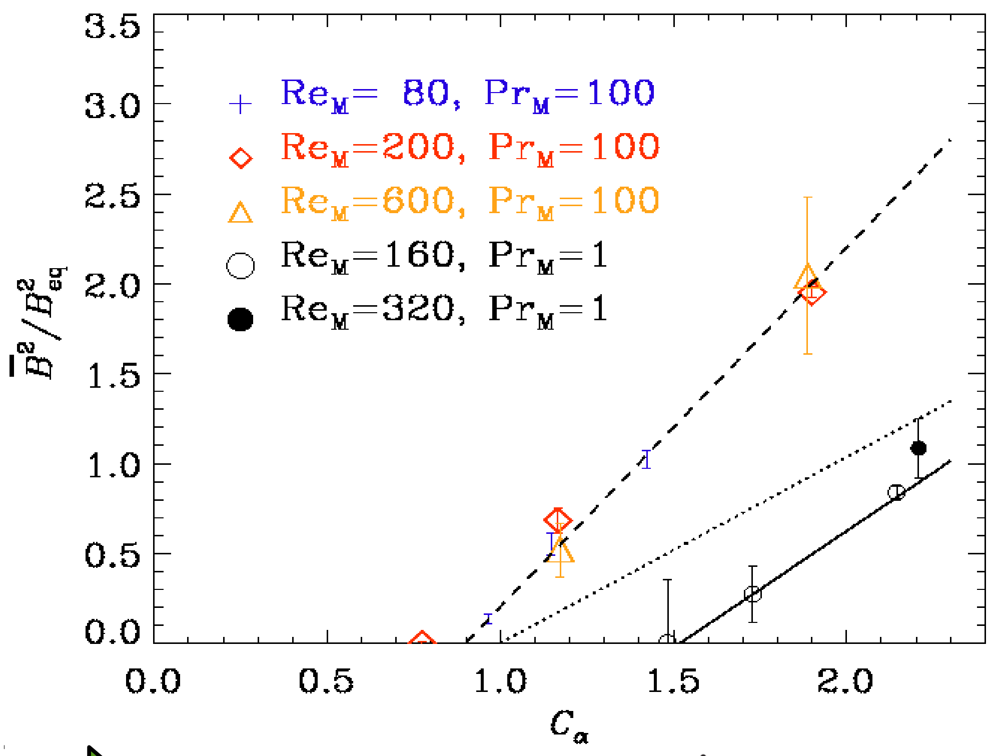
Predictions: $\overline{B}^2_{sat}/B^2_{eq} = \epsilon_f \left(\frac{k_f}{k_m}\right) - \iota$ $\qquad \epsilon_f^{crit} = \iota \epsilon_m \left(\frac{k_f}{k_m}\right)^{-1}$

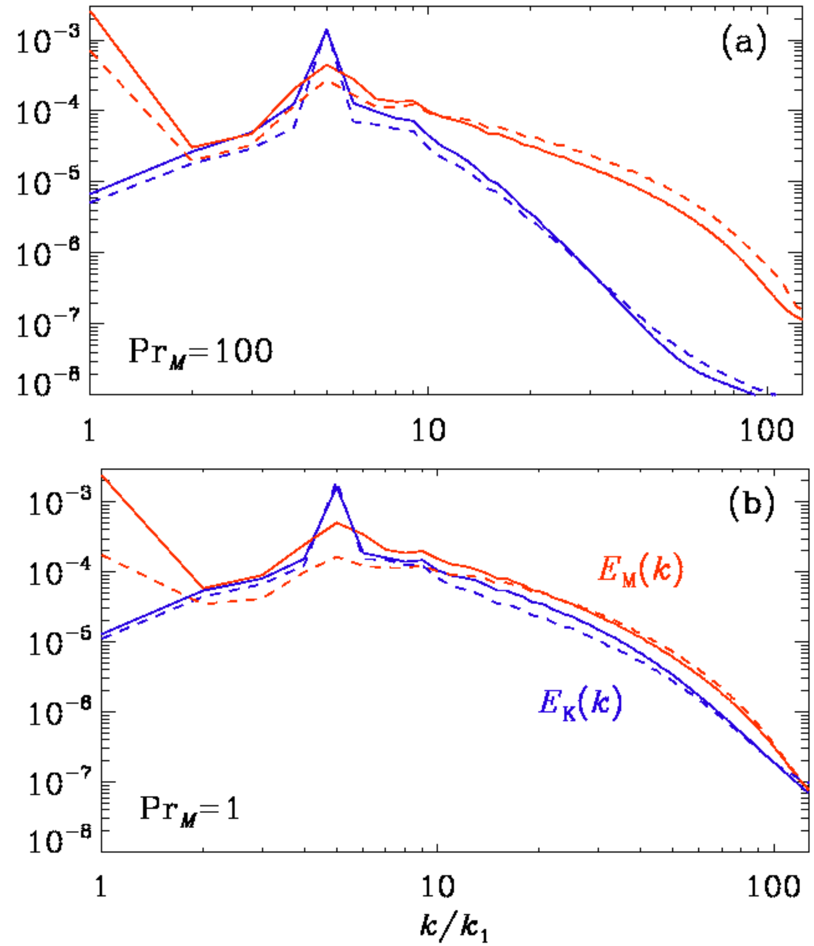
NB: $k_1 = k_m$

High $\mathrm{Re_M}$

$\mathrm{Re_M} \to 2000$ *(Pietarila Graham, et. al. 2012)*

Match their parameters.





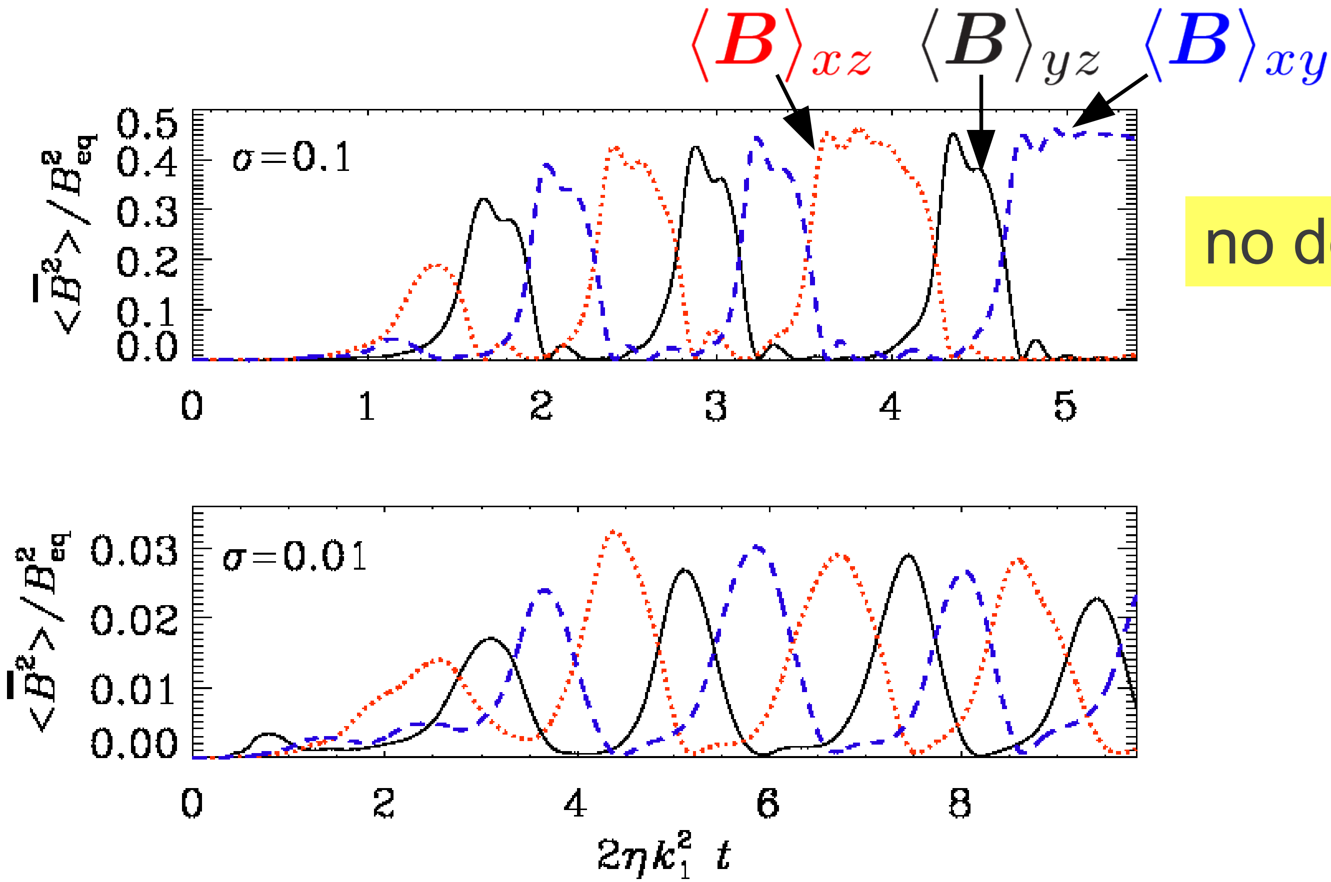
No change in C_α^{crit} for high $\mathrm{Re_M}$. *(Candelaresi, Brandenburg 2012)*

B_{eq} is underestimated for high $\mathrm{Pr_M}$ due to viscous losses.

ABC-Flow Forcing

Forcing: $$\boldsymbol{f}(\boldsymbol{x},t) = \frac{f_0}{\sqrt{\frac{3}{2}(1+\sigma^2)}} \begin{pmatrix} \sin(X_3)+\sigma\cos(X_2) \\ \sin(X_1)+\sigma\cos(X_3) \\ \sin(X_2)+\sigma\cos(X_1) \end{pmatrix}$$

$$X_i = k_\mathrm{f} x_i + \theta_0 \cos(\omega_i t)$$



no dominant mode

Summary

- MF prediction reproduced in DNS.
- Discrepancy of (Graham) due to SSD contamination.
- ABC-flow produces oscillating modes.

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