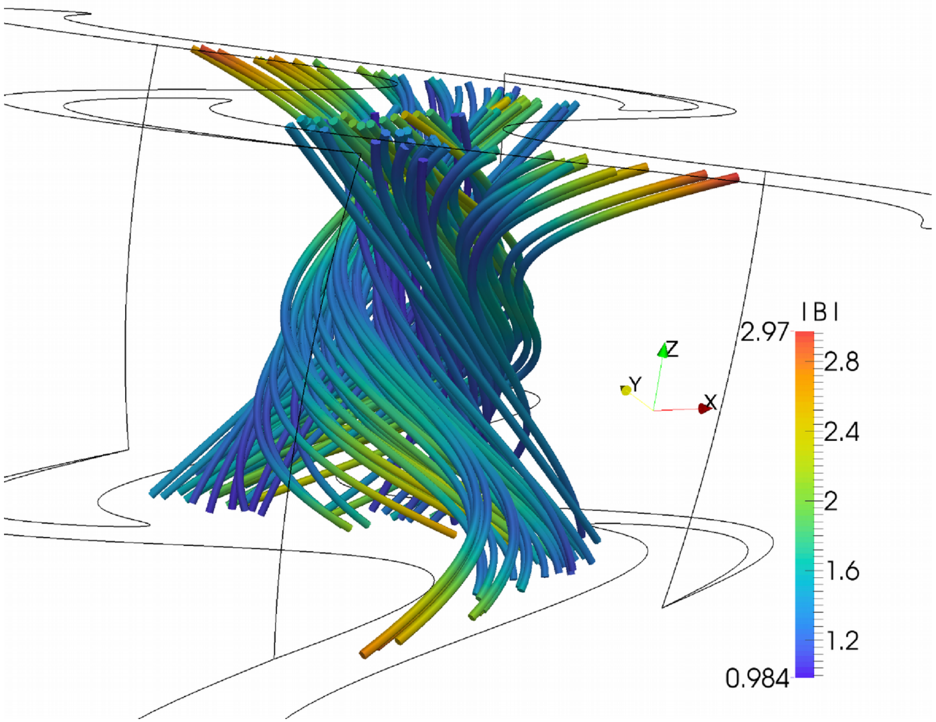
# Mimetic Methods for Magnetic Energy Minimisation

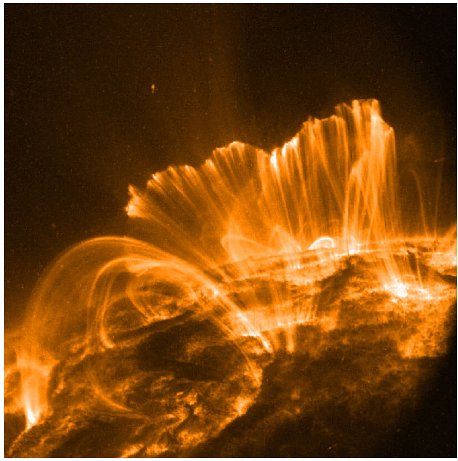
Simon Candelaresi



# Force-Free Magnetic Fields

Solar corona: low plasma beta and magnetic resistivity

NASA



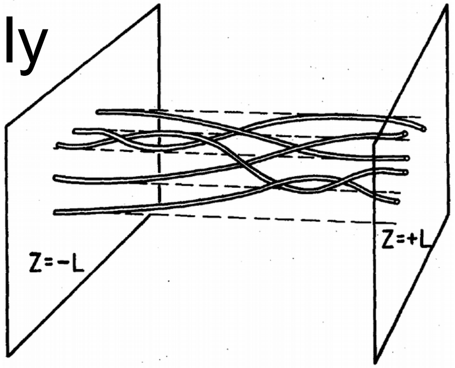
⇨ Force-free magnetic fields

⇨ Minimum energy state

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \;\Leftrightarrow\; \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

**Parker:** Equilibrium with the same topology exists only if the twist varies uniformly along the field lines. Strongly braided fields → topological dissipation.

*(Parker 1972)*




Braided fields from foot point motion complex enough. *(Parker 1983)*

Solutions possible with filamentary current structures (sheets).

*(Mikic 1989, Low 2010)*

# Ideal Field Relaxation

Ideal induction eq.: $$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

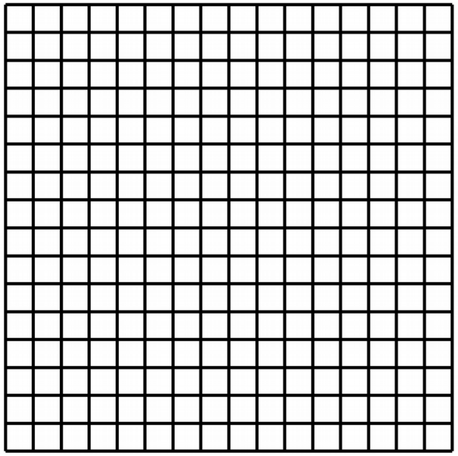
Frozen in magnetic field.

*(Batchelor, 1950)*

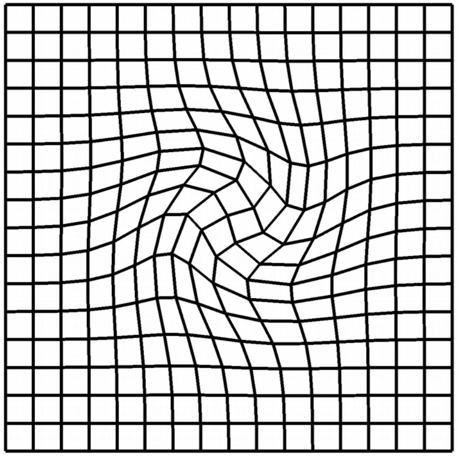
**But**: Numerical diffusion in finite difference Eulerian codes.

**Solution**: Lagrangian description of moving fluid particles:

$$\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$$



$$\mathbf{x}(\mathbf{X}, t)$$



# Ideal Field Relaxation

Field evolution: $$B_i(\mathbf{X},t) = \frac{1}{\Delta}\sum_{j=1}^{3}\frac{\partial x_i}{\partial X_j}B_j(\mathbf{X},0)$$

$$\Delta = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

Preserves topology and divergence-freeness.

Grid evolution: $$\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \mathbf{u}\left(\mathbf{x}(\mathbf{X},t),t\right)$$

Magneto-frictional term: $\mathbf{u} = \gamma \mathbf{J}\times\mathbf{B}$ $\qquad$ $\mathbf{J} = \nabla\times\mathbf{B}$

$$\Rightarrow \frac{\mathrm{d}E_{\mathrm{M}}}{\mathrm{d}t} < 0$$

*(Craig and Sneyd 1986)*

# Numerical Curl Operator

Compute $\mathbf{J} = \nabla \times \mathbf{B}$ on a distorted grid:

$$\frac{\partial B_i}{\partial x_j} = X_{\alpha,j}(x_{i,\alpha\beta} B^0_\beta \Delta^{-1} + x_{i,\beta} B^0_{\beta,\alpha} \Delta^{-1} - x_{i,\beta} B^0_\beta \Delta^{-2} \Delta_{,\alpha})$$

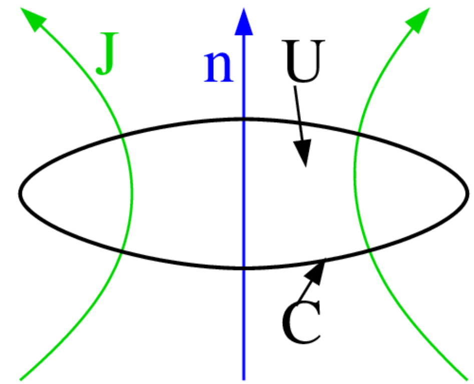
$$B^0_i = B_i(0)$$

*(Craig and Sneyd 1986)*

- Multiplication of several terms leads to high numerical errors.
- Current not divergence free: $\nabla \cdot \mathbf{J} \neq 0$
- Only reaching a certain force-freeness. *(Pontin et al. 2009)*

# Mimetic Numerical Operators

$$I = \int_U \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}S = \oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r}$$




Discretized:

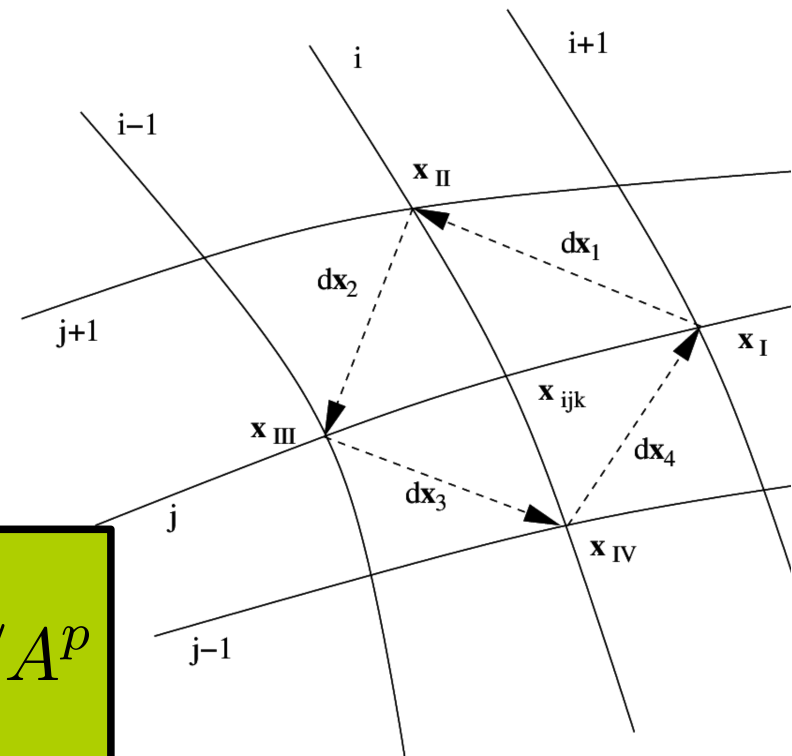
$$I \approx \mathbf{J}(\mathbf{X}_{ijk})\cdot\mathbf{n}A = \sum_{r=1}^{4}\mathbf{B}_r\cdot\mathrm{d}\mathbf{x}_r$$

$$\mathbf{J}(\mathbf{X}_U) \approx \mathbf{J}(\mathbf{X}_{\mathbf{ijk}}), \quad \mathbf{X}_U \in U$$

3 planes will give 3 l.i. normal vectors:

$$I^p = \mathbf{J}(\mathbf{X}_{ijk})\cdot\mathbf{n}^p = \sum_{r=1}^{4}\mathbf{B}_r^p\cdot\mathrm{d}\mathbf{x}_r/A^p$$

Inversion yields $\mathbf{J}$ with $\nabla\cdot\mathbf{J} = 0.$




*(Hyman, Shashkov 1997)*

# Methods

Ideal (non-resistive) evolution
Frozen in magnetic field → use Lagrangian method

*(Batchelor, 1950)*

Preserves topology and divergence-freeness.

Magneto-frictional term: $\mathbf{u} = \gamma \mathbf{J} \times \mathbf{B}$ $\quad \mathbf{J} = \nabla \times \mathbf{B}$

$$\Rightarrow \frac{\mathrm{d}E_\mathrm{M}}{\mathrm{d}t} < 0$$ *(Craig and Sneyd 1986)*

Fluid with pressure: $\mathbf{u} = \mathbf{J} \times \mathbf{B} - \beta \nabla \rho$

Fluid with inertia: $\mathrm{d}\mathbf{u}/\mathrm{d}t = (\mathbf{J} \times \mathbf{B} - \nu \mathbf{u} - \beta \nabla \rho)/\rho$

For $\mathbf{J} = \nabla \times \mathbf{B}$ use mimetic numerical operators.

*(Hyman, Shashkov 1997)*

Own GPU code GLEMuR: (https://github.com/SimonCan/glemur)

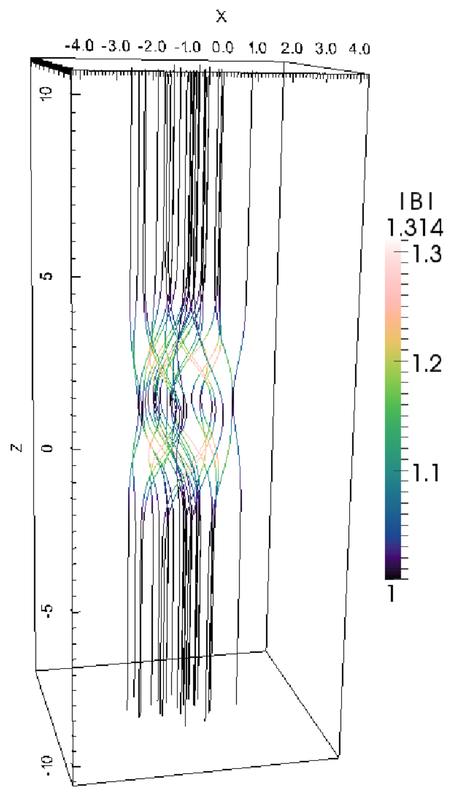
*(Candelaresi et al. 2014)*

# Simulations

- GPU code GLEMuR (**G**pu-based **L**agrangian mim**E**tic **M**agnetic **R**elaxation)
- line tied boundaries
- mimetic vs. classic

*(Candelaresi et al. 2014)*

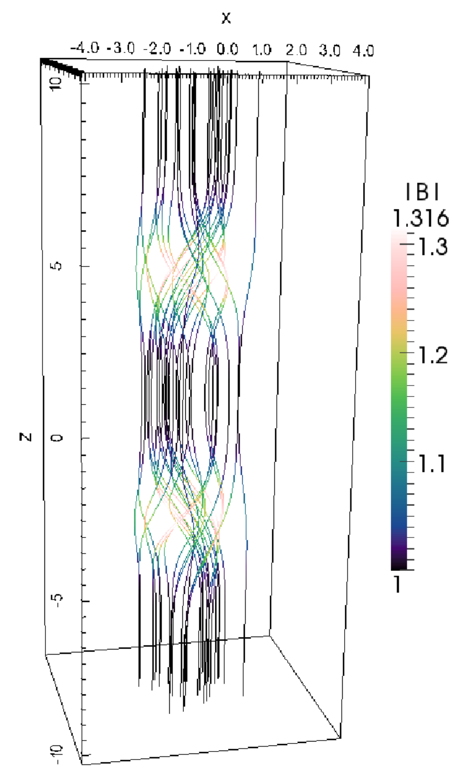
Nvidia Tesla K40




we know:

$$\lim_{t\to\infty} \mathbf{B}(t)$$

$$\lim_{t\to\infty} \mathbf{x}(t)$$




we know:

$$\lim_{t\to\infty} \mathbf{B}(t)$$

# Quality Parameters

Deviation from the expected relaxed state:

$$\sigma_{\mathbf{x}} = \sqrt{\frac{1}{N}\sum_{ijk}\left(\mathbf{x}(\mathbf{X}_{ijk}) - \mathbf{x}_{\mathrm{relax}}(\mathbf{X}_{ijk})\right)^2}$$

$$\sigma_{\mathbf{B}} = \sqrt{\frac{1}{N}\sum_{ijk}\left(\mathbf{B}(\mathbf{X}_{ijk}) - \mathbf{B}_{\mathrm{relax}}(\mathbf{X}_{ijk})\right)^2}$$

Free magnetic energy:

$$E_{\mathrm{M}}^{\mathrm{free}} = E_{\mathrm{M}} - E_{\mathrm{M}}^{\mathrm{bkg}}$$

$$E_{\mathrm{M}} = \int_V \mathbf{B}^2/2 \, \mathrm{d}V \qquad \mathbf{B}^{\mathrm{bkg}} = B_0 \hat{e}_z$$

# Quality Parameters

For a force-free field: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

⇨ Force-free parameter does not change along field lines.

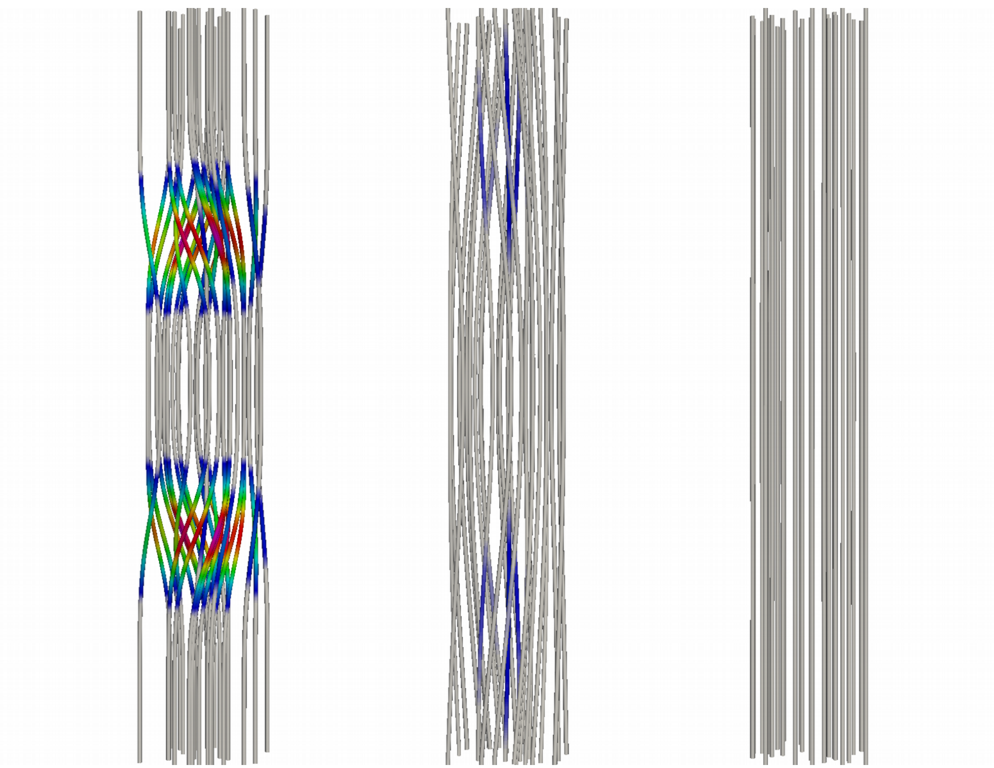
⇨ Measure the change of $\alpha^* = \dfrac{\mathbf{J} \cdot \mathbf{B}}{\mathbf{B}^2}$ along field lines:

$$\epsilon^* = \max_{i,j} \left( a_r \frac{\alpha^*(\mathbf{X}_i) - \alpha^*(\mathbf{X}_j)}{|\mathbf{X}_i - \mathbf{X}_j|} \right); \quad \mathbf{X}_i, \mathbf{X}_j \in s_\alpha$$

Particular field line: $s_\alpha = \{(0, 0, Z) : Z \in [-L_z/2, L_z/2]\}$

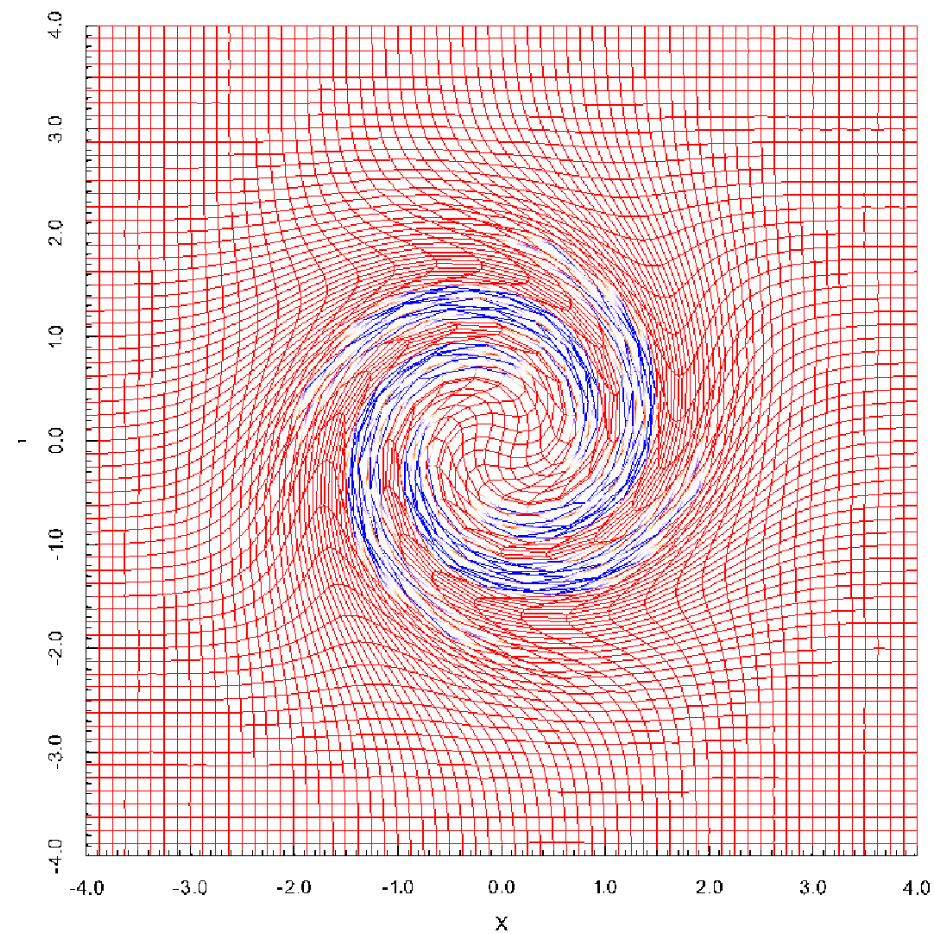
# Field Relaxation

Magnetic streamlines:



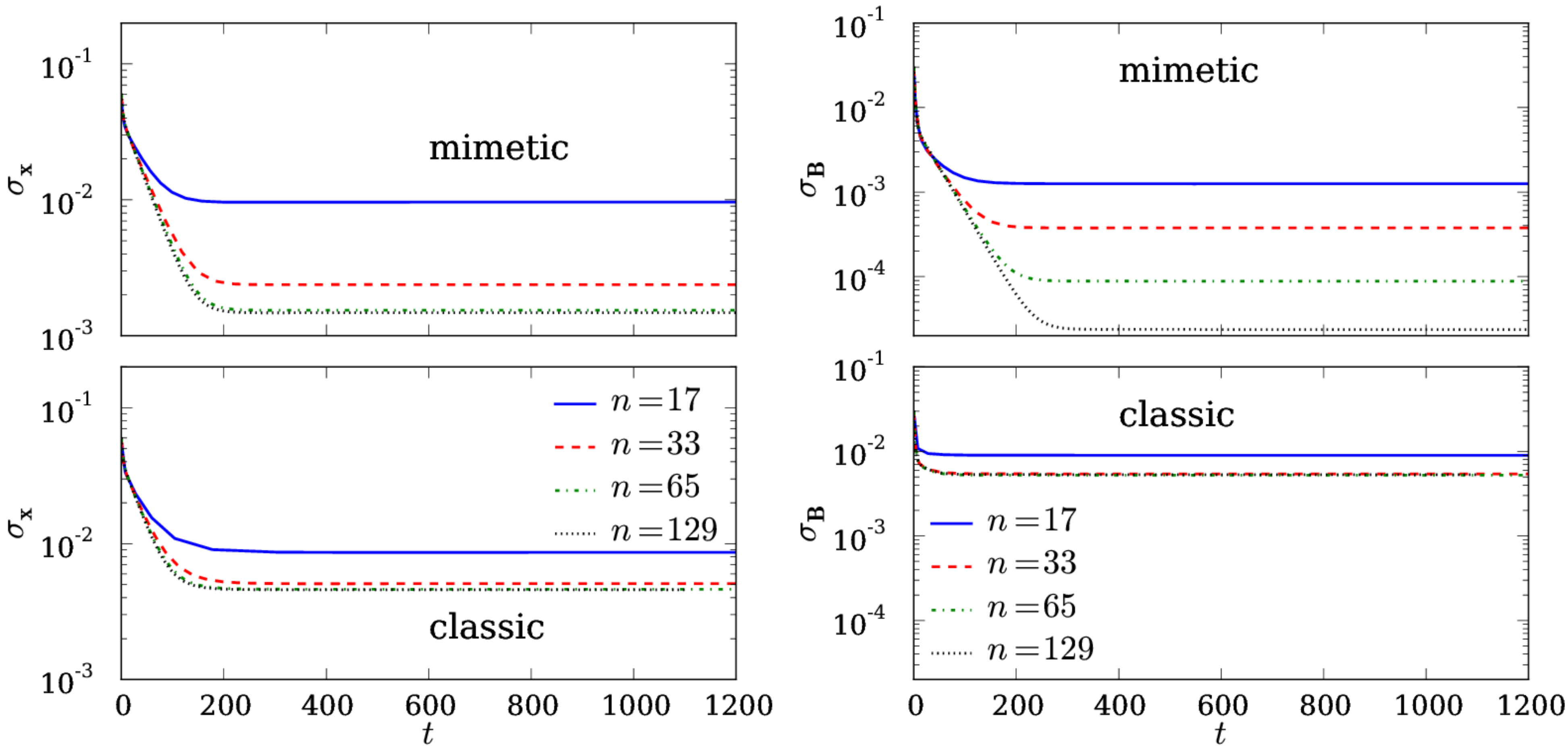
$t = 0$ $t = 10$ $t = 430$

Grid distortion at mid-plane:



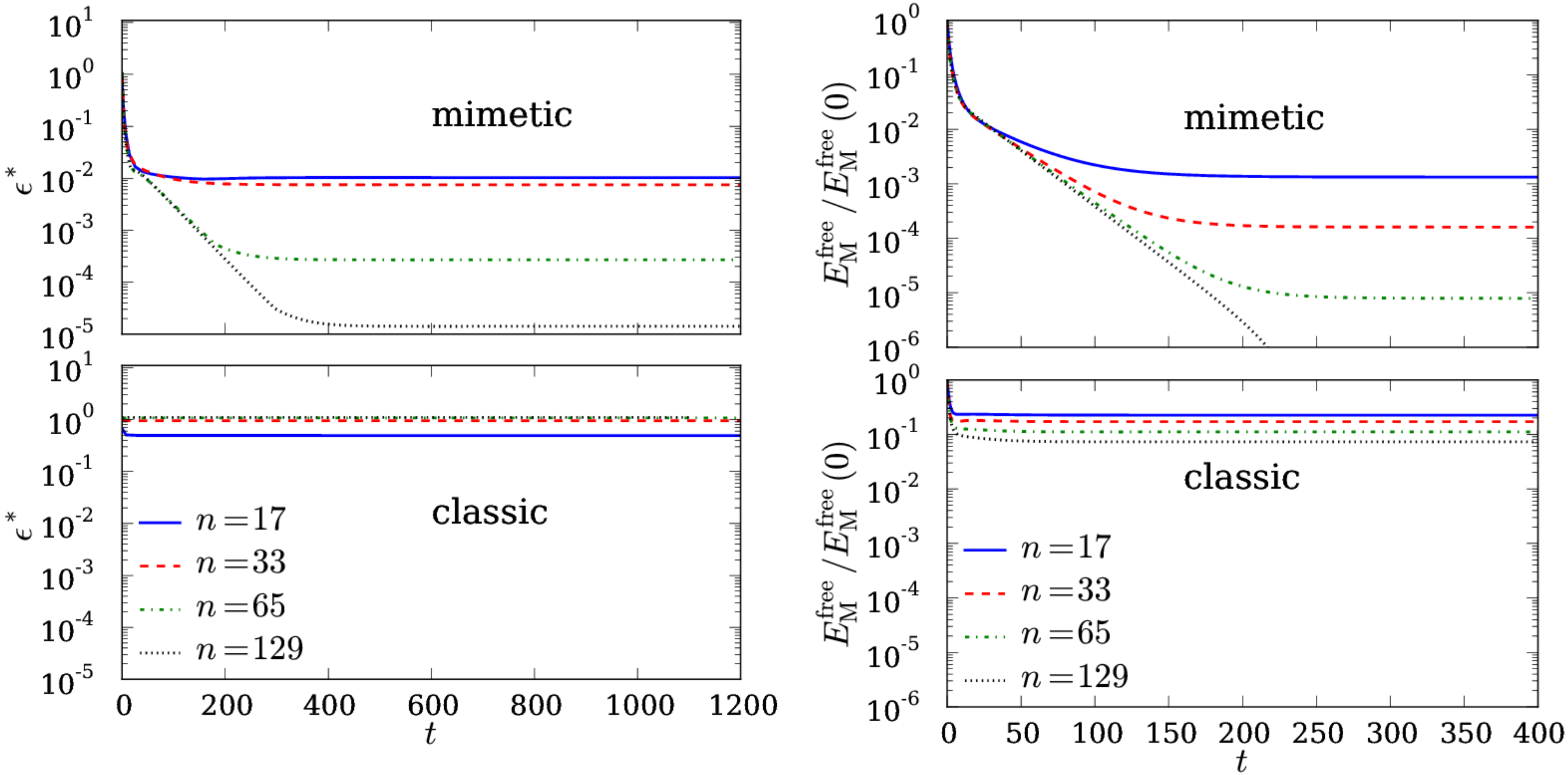

movie

# Relaxation Quality



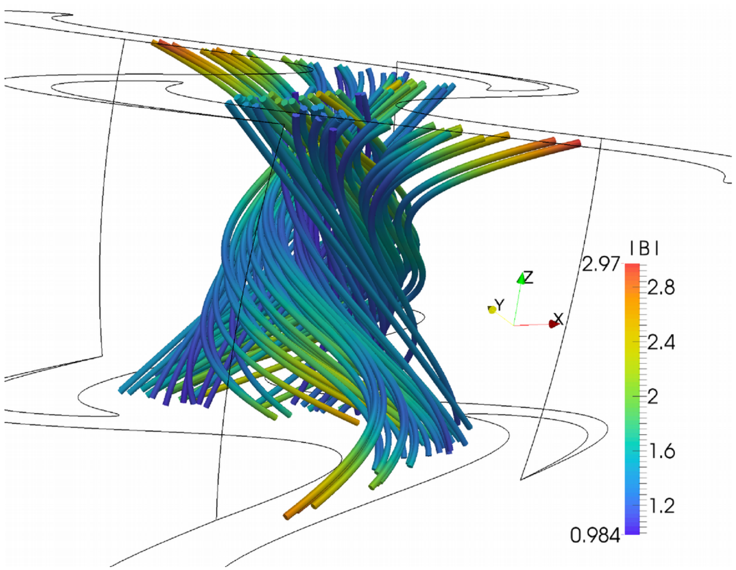

Closer to the analytical solution by 3 orders of magnitude.

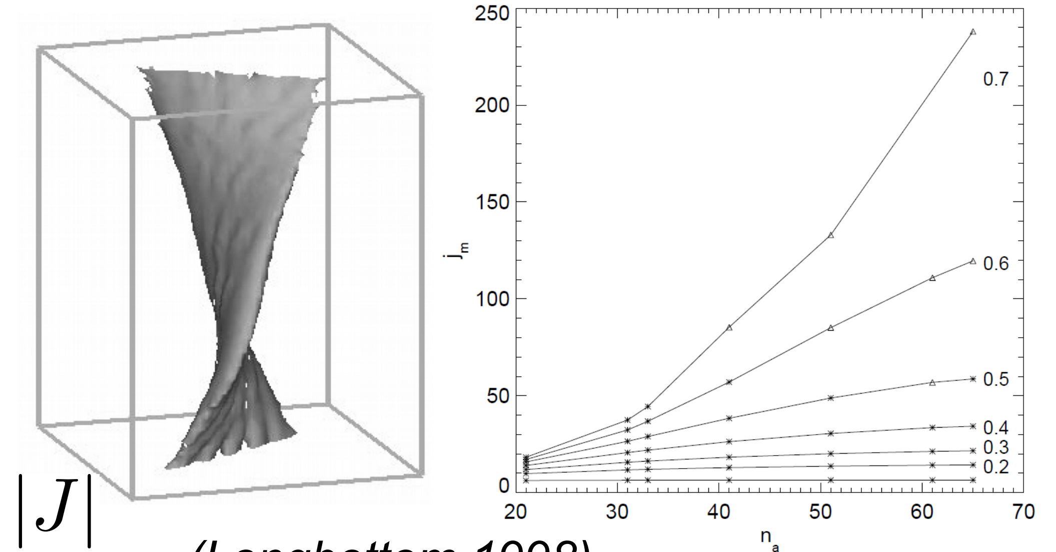
# Relaxation Quality



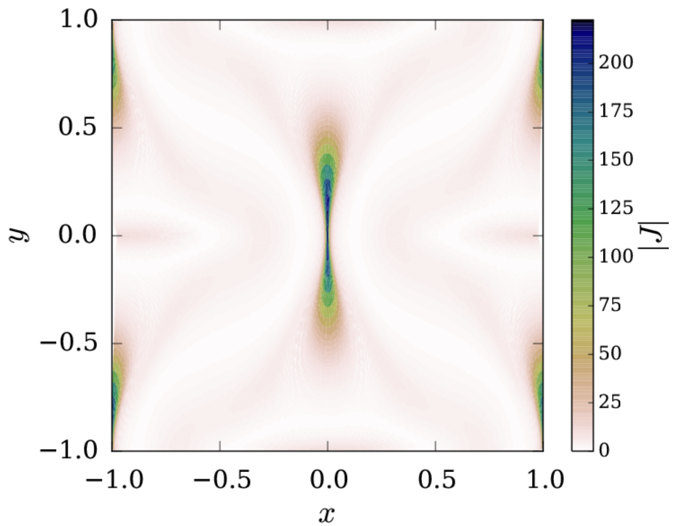

Closer to force-free state by 5 orders of magnitude.

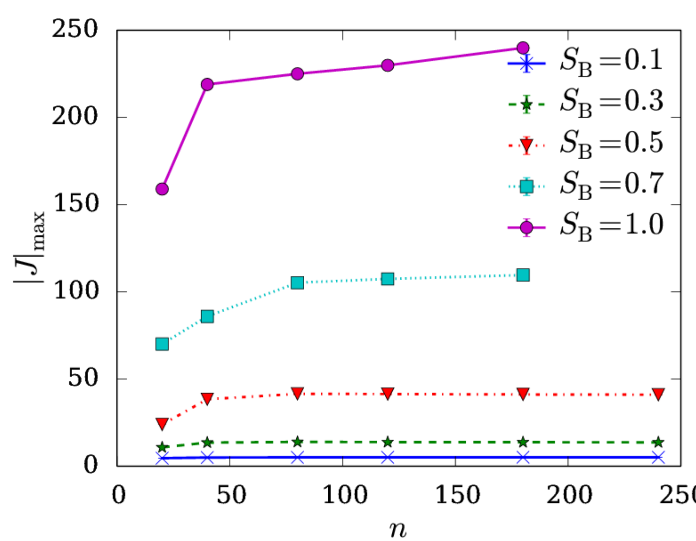
# Distorted Magnetic Fields




(Longbottom 1998)

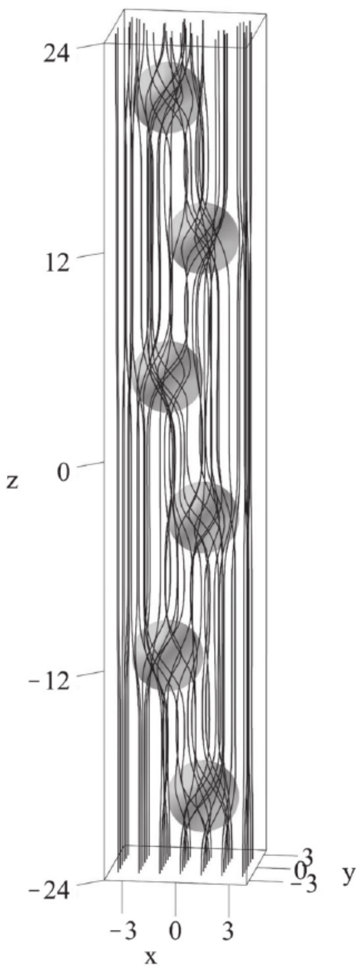


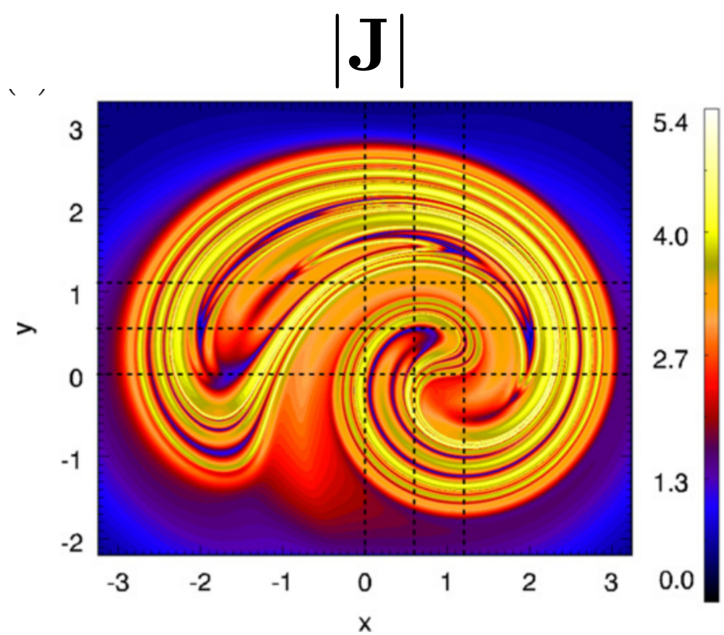



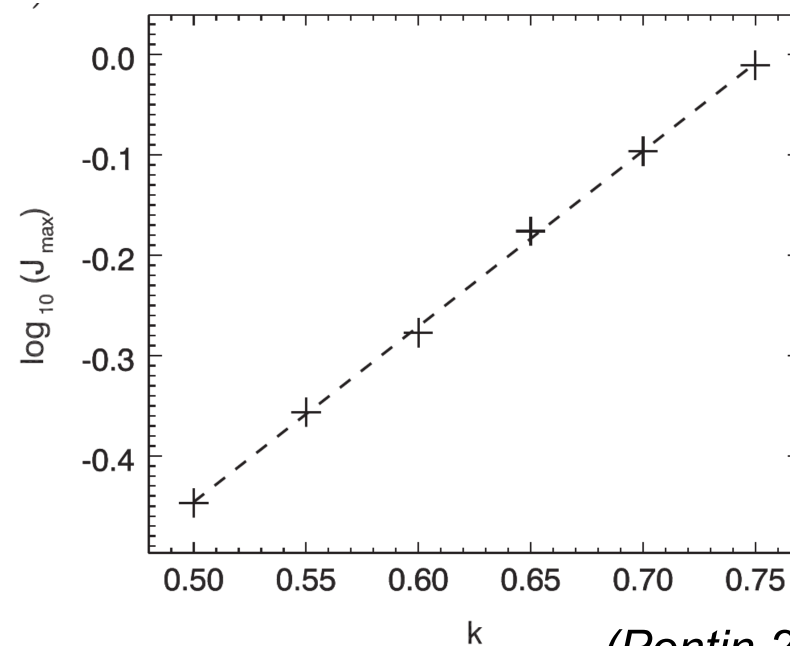

- resolved current concentrations
- shear leads to strong currents

(Candelaresi et al. 2015)

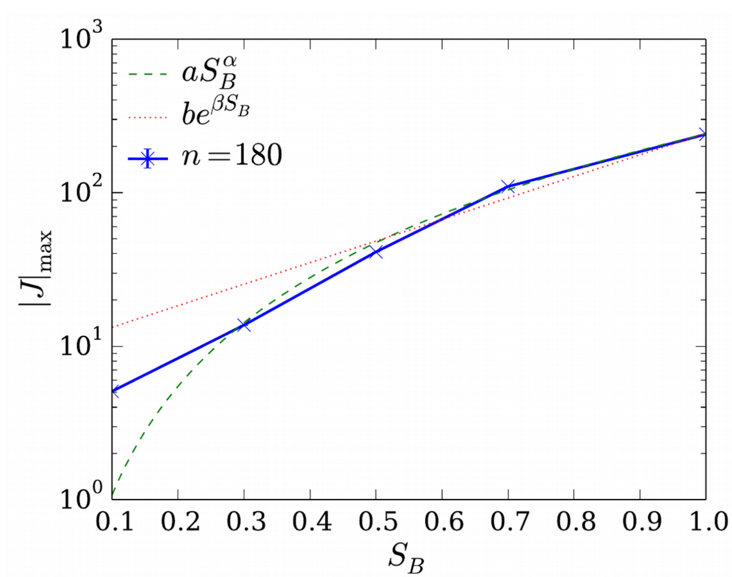
# Exponential Increase in Current




(Pontin 2015)




(Candelaresi 2015)

- lengths decrease exp. with complexity
- current increases exp. with complexity

# Conclusions

- Topology preserving relaxation of magnetic fields.
- Mimetic methods more capable of producing force-free fields.
- GLEMuR code running on GPUs.
- Current concentrations not singular.
- Current increases strongly with field complexity.

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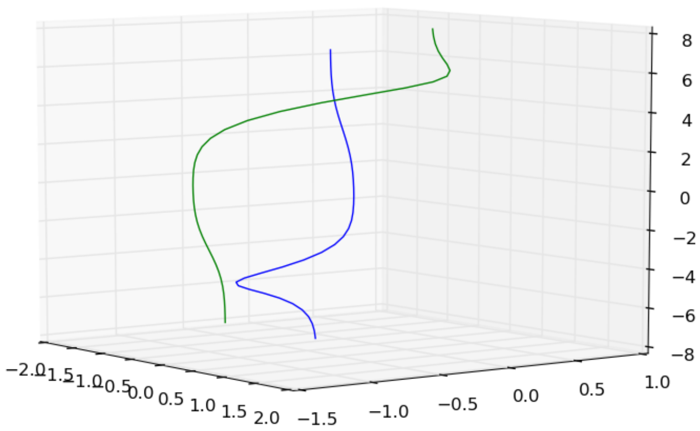
# Code Details

- written in C++
- running on GPUs
- VTK data format
- $6^{th}$ order Runge-Kutta time stepping
- periodic and line-tied boundaries
- post processing routines in Python

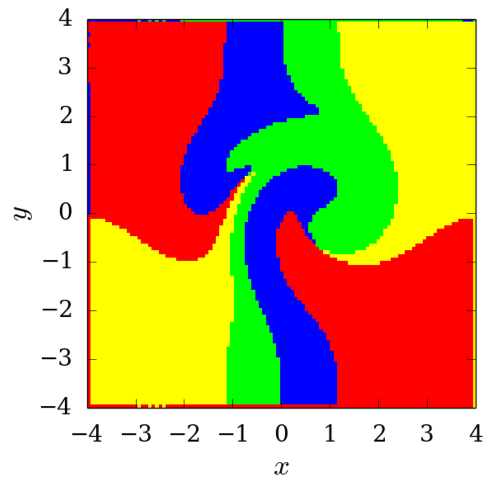
```
// compute the norm of JxB/B**2
__global__ void JxB_B2(REAL *B, REAL *J, REAL *JxB_B2, int dimX, int dimY, int dimZ) {
        int i = threadIdx.x + blockDim.x * blockIdx.x;
        int j = threadIdx.y + blockDim.y * blockIdx.y;
        int k = threadIdx.z + blockDim.z * blockIdx.z;
        int p = threadIdx.x;
        int q = threadIdx.y;
        int r = threadIdx.z;
        int l;
        REAL B2;

        // shared memory for faster communication, the size is assigned dynamically
        extern __shared__ REAL s[];
        REAL *Bs = s;                               // magnetic field
        REAL *Js = &s[3 * dimX * dimY * dimZ];      // electric current density
        REAL *JxBs = &Js[3 * dimX * dimY * dimZ];   // JxB

        // copy from global memory into shared memory
        if ((i < dev_p.nx) && (j < dev_p.ny) && (k < dev_p.nz)) {
                for (l = 0; l < 3; l++) {
                        Bs[l + p*3 + q*dimX*3 + r*dimX*dimY*3] = B[l + (i+1)*3 + (j+1)*(dev_p.nx+2)*3 + (k+1)*(dev_p.nx+2)*(dev_p.ny+2)*3];
                        Js[l + p*3 + q*dimX*3 + r*dimX*dimY*3] = J[l + i*3 + j*dev_p.nx*3 + k*dev_p.nx*dev_p.ny*3];
                }

                cross(&Js[0 + p*3 + q*dimX*3 + r*dimX*dimY*3],
                          &Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3],
                          &JxBs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3]);

                B2 = dot(&Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3], &Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3]);

                // return result into global memory
                JxB_B2[i + j*dev_p.nx + k*dev_p.nx*dev_p.ny] = norm(&JxBs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3])/B2;
        }
}
```

# Post-Processing

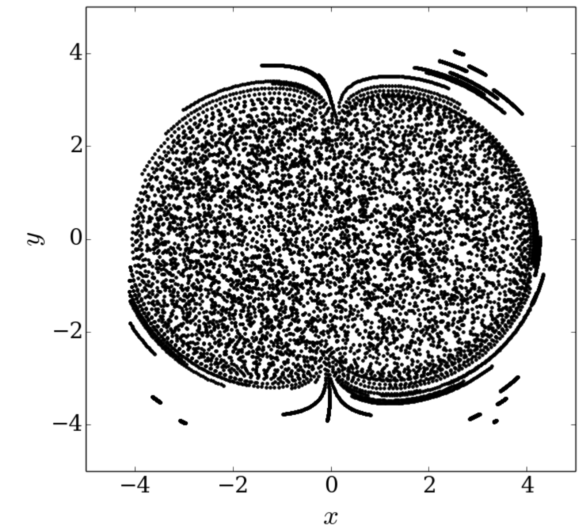
streamlines



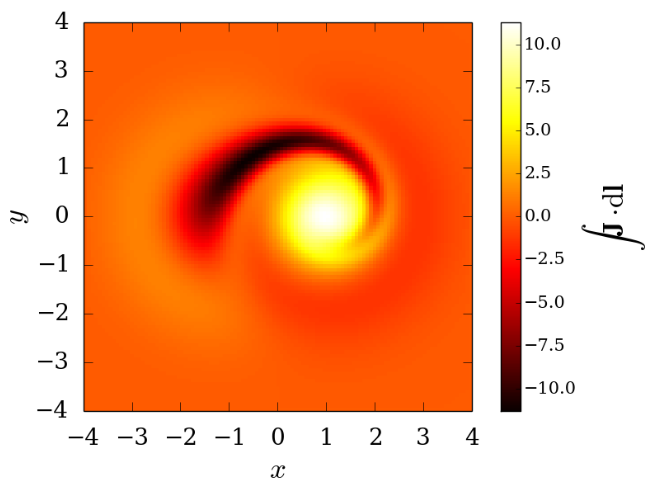
field line mapping




Poincaré maps




line integration




save and read as vtk file

```
s0 = gm.streamInit(tol = 0.01)
```

⇩

```
stream.vtk
```

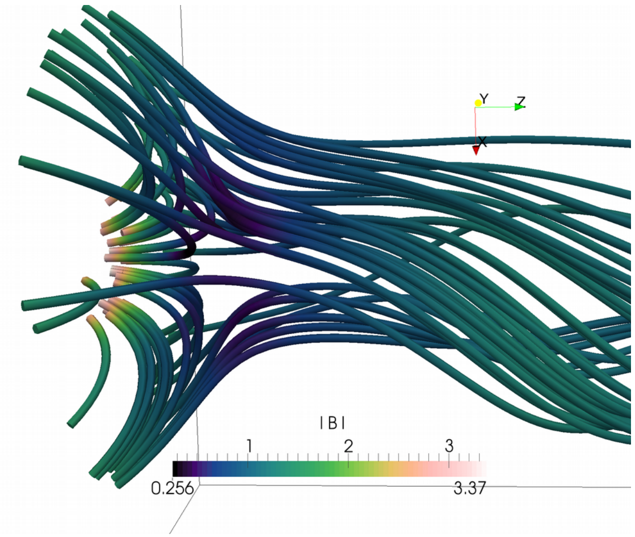
⇩

```
sr = gm.readStream()
```

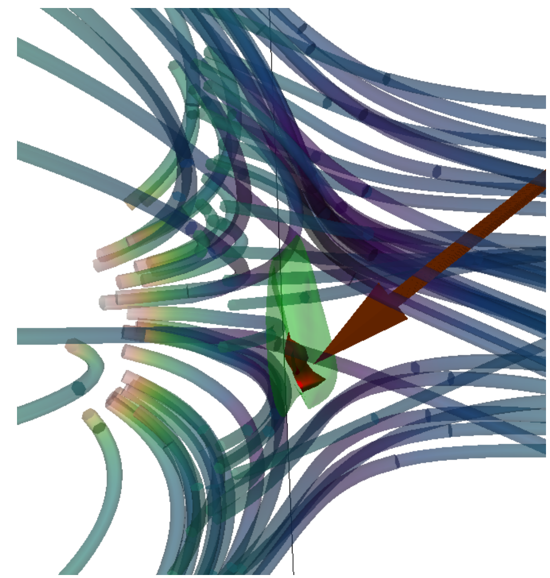
# Magnetic Nulls

Singular current sheets observed at magnetic nulls (B = 0)

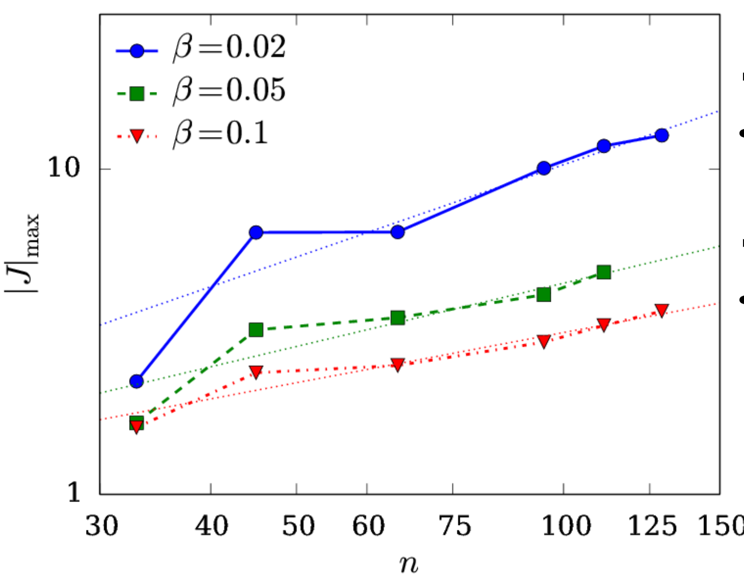
*(Syrovatskiĭ 1971; Pontin & Craig 2005; Fuentes-Fernández & Parnell 2012, 2013; Craig & Pontin 2014)*




$$\mathbf{u} = \mathbf{J} \times \mathbf{B}$$

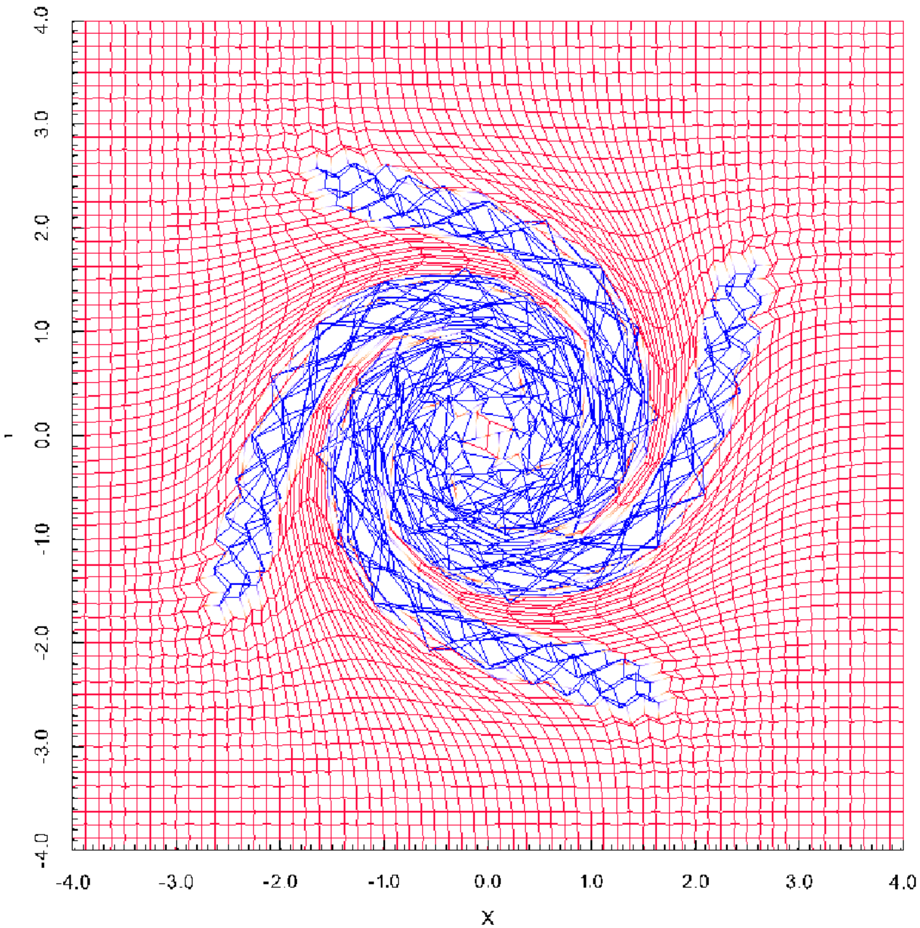


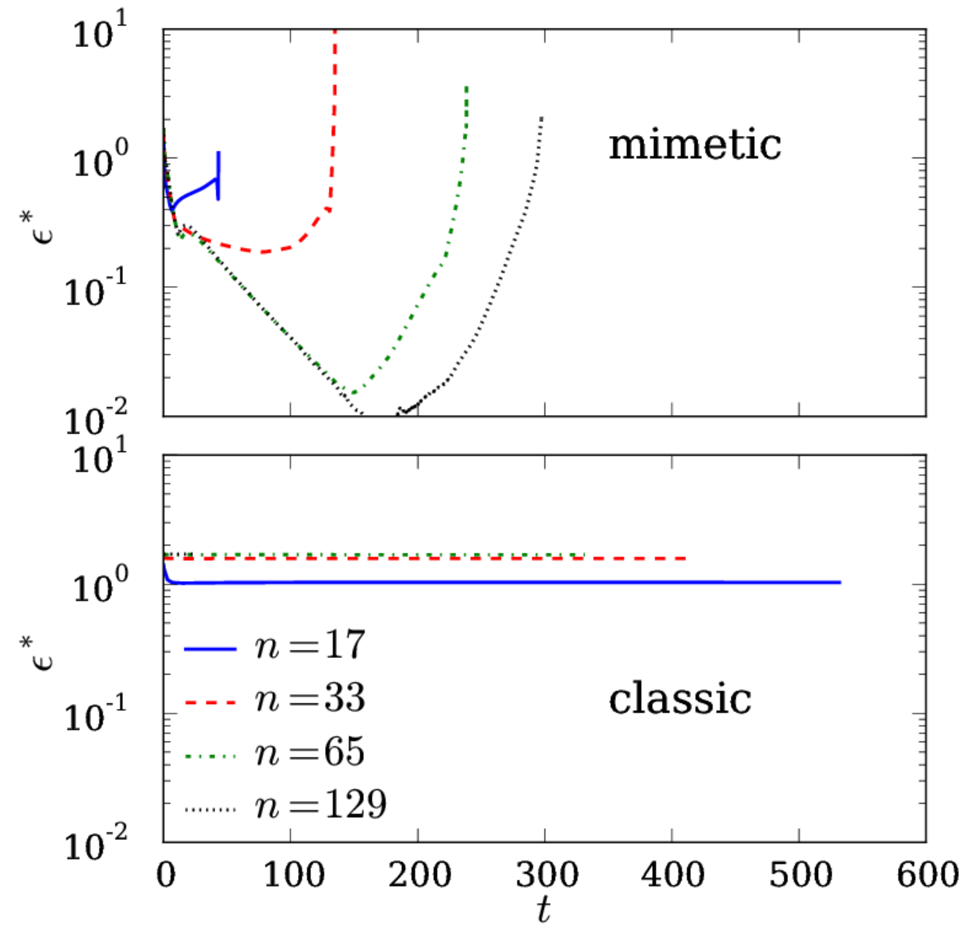
$$\mathbf{u} = \mathbf{J} \times \mathbf{B} - \beta \nabla \rho$$




- singular current sheets at magnetic nulls
- Pressure cannot balance singularity.

# Limitations




red: convex
blue: concave

For concave cells the method becomes unstable.
**But**: results before crash better than classic method.