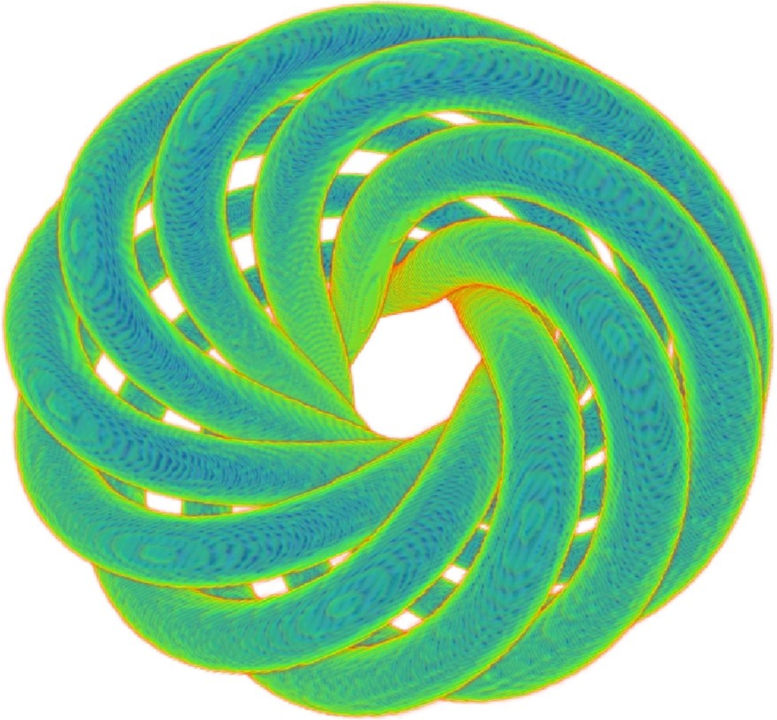
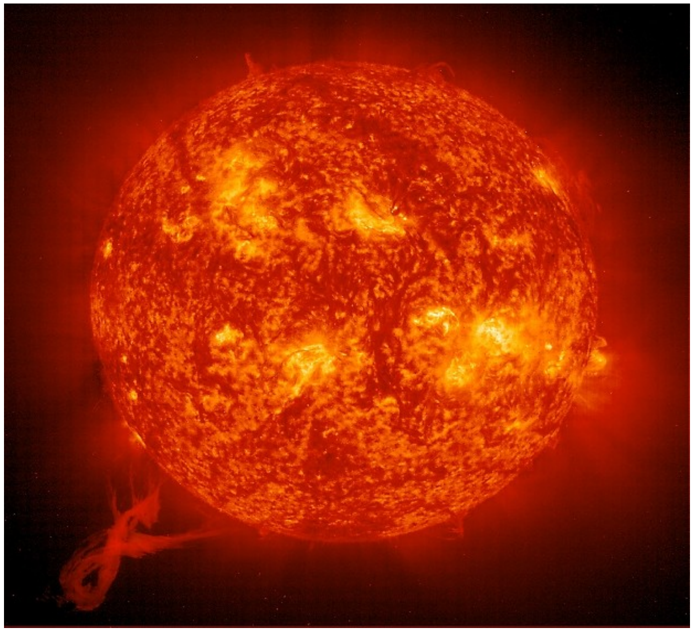
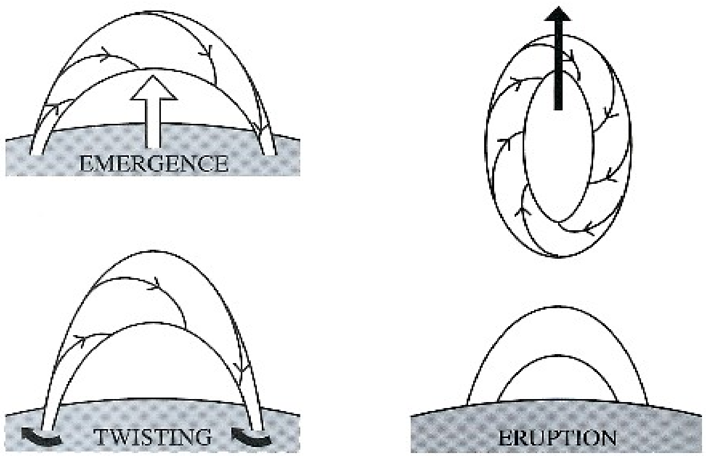
Topological aspects in magnetic field dynamics

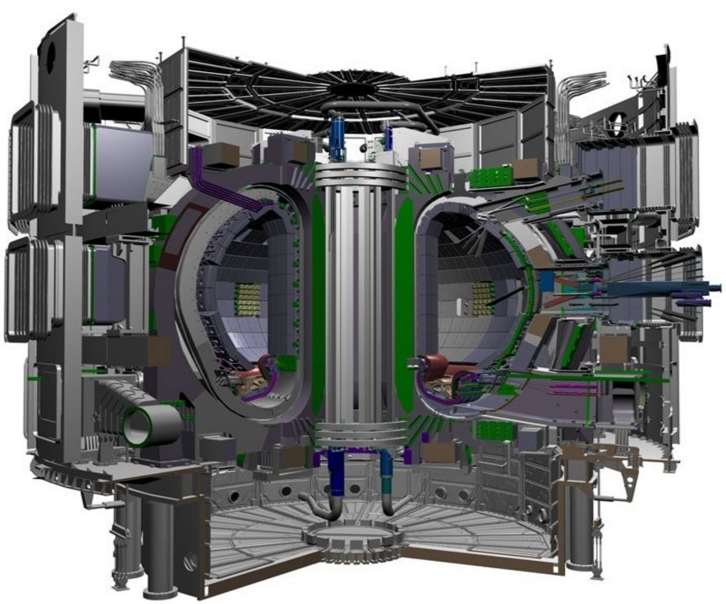
Simon Candelaresi



Twisted magnetic fields







- Twisted fields are more likely to erupt (Canfield et al. 1999).
- Twist increases the stability of magnetic fields in tokamaks.

Magnetic helicity

$$H_\mathrm{M} = \int_V \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V = 2n\phi_1\phi_2$$

$$\phi_i = \int_{S_i} \mathbf{B}\cdot\mathrm{d}\mathbf{S}$$

Realizability condition:

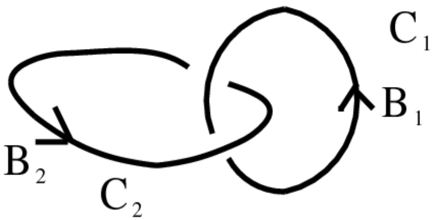
$$E_\mathrm{m}(k) \geq k|H(k)|/2\mu_0$$

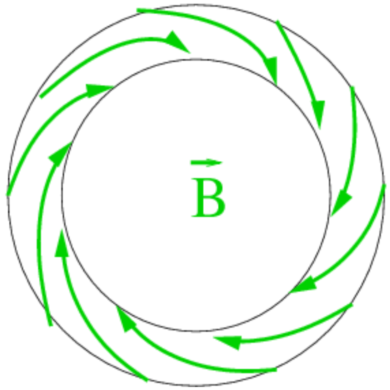
Magnetic energy is bound from below by magnetic helicity.

magnetic helicity conservation

$$\mathrm{Re_M} \to \infty$$

$$\frac{\mathrm{d}H_\mathrm{M}}{\mathrm{d}t} = 0$$





twisted field


trefoil knot

Stability criteria

Ideal MHD: $\mu = 0$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{U} \times \mathbf{B})$$

Induction equation: $\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{U} \times \mathbf{B})$

	constraint	equilibrium
Woltjer (1958):	$\frac{\partial}{\partial t} \int_V \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \mathbf{B} = \alpha \mathbf{B}$
Taylor (1974):	$\frac{\partial}{\partial t} \int_{\tilde{V}} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \mathbf{B} = \alpha(a, b) \mathbf{B}$ (constant along field line)

V total volume $\tilde{V}$ volume along magnetic field line

Creation of magnetic field and magnetic helicity

Mean-field decomposition: $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{b}$

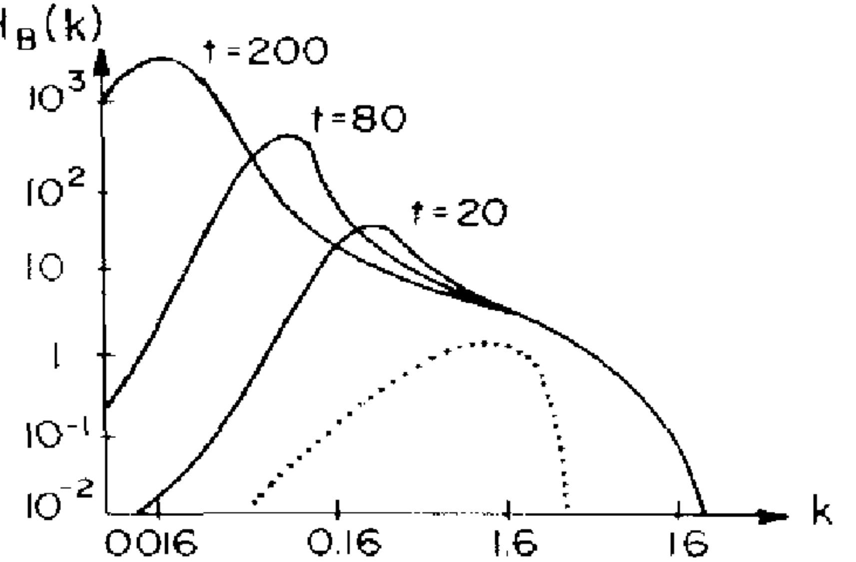
Induction equation: $\partial_t \overline{\mathbf{B}} = \eta \nabla^2 \overline{\mathbf{B}} + \boldsymbol{\nabla} \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \overline{\boldsymbol{\mathcal{E}}})$

Electromotive force: $\overline{\boldsymbol{\mathcal{E}}} \underset{\alpha}{=} \overline{\mathbf{u} \times \mathbf{b}} = \alpha \overline{\mathbf{B}} - \eta_{\rm t} \boldsymbol{\nabla} \times \overline{\mathbf{B}}$

α effect: $\alpha = \alpha_{\rm K} + \alpha_{\rm M} = -\tau \overline{\boldsymbol{\omega} \cdot \mathbf{u}}/3 + \overline{\mathbf{j} \cdot \mathbf{b}}/(3\overline{\rho})$

Inverse cascade:

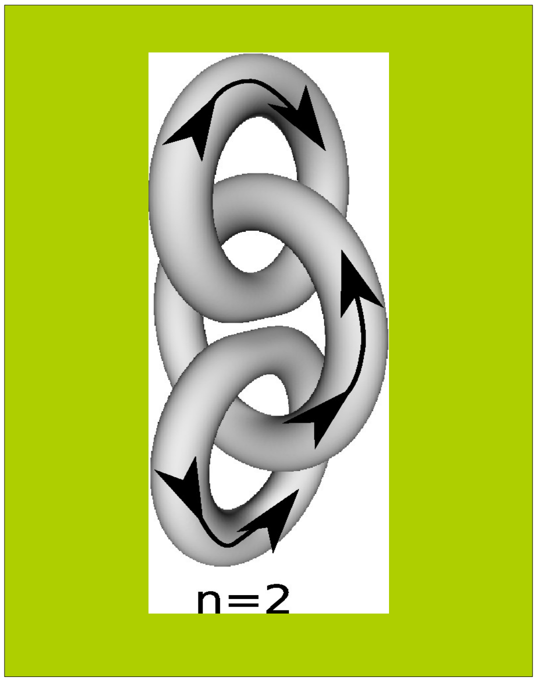
Large- and small-scale magnetic helicity of opposite sign is created.



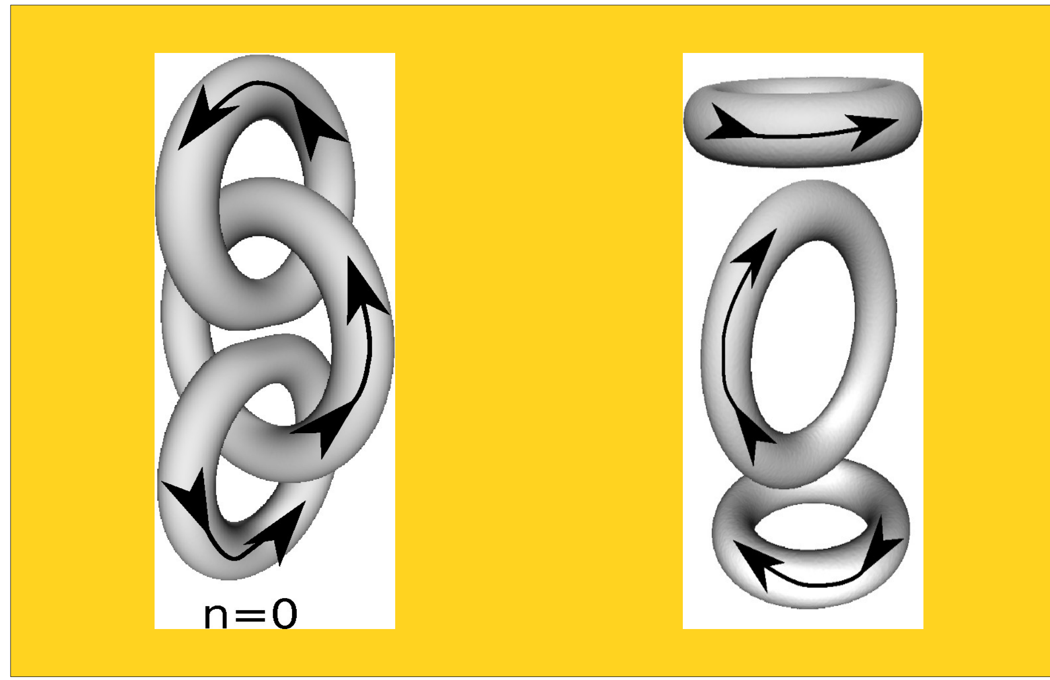
Leorat et al., 1975

Interlocked flux rings

$H_{\mathrm{M}} \neq 0$



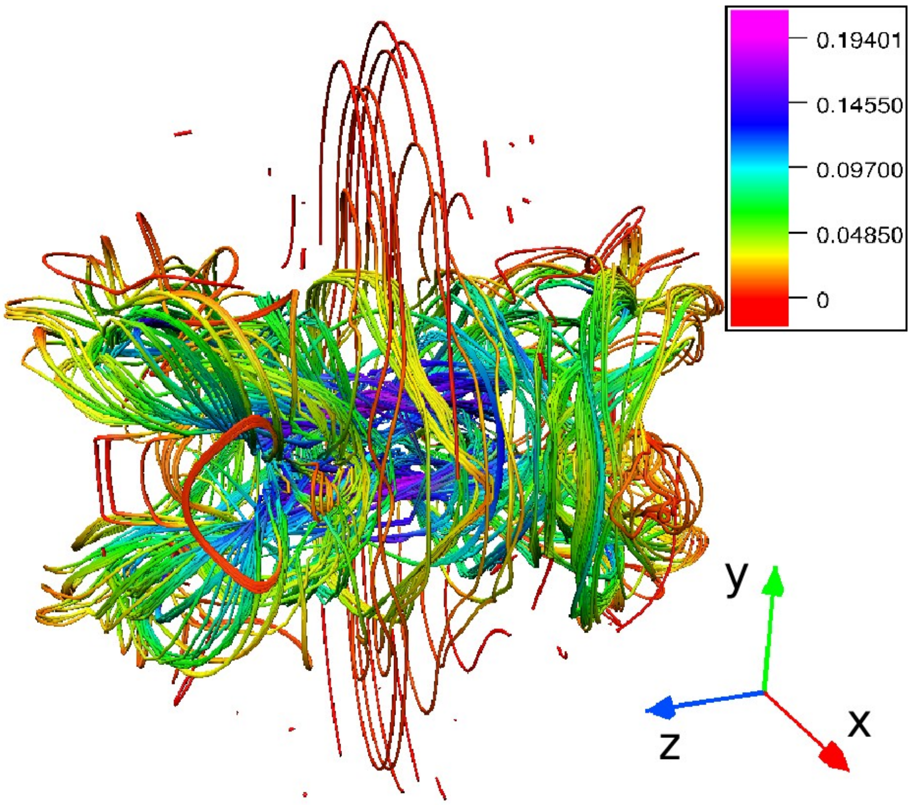
$H_{\mathrm{M}} = 0$



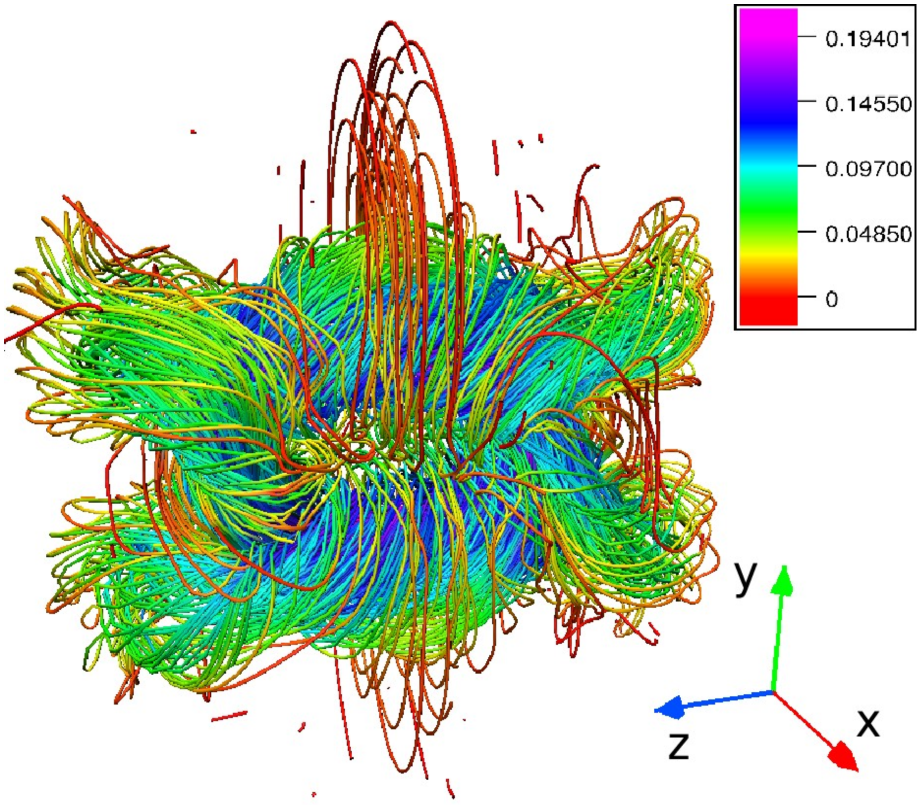
- Isothermal compressible gas
- Viscous medium
- Periodic boundaries

Interlocked flux rings

$\tau = 4$

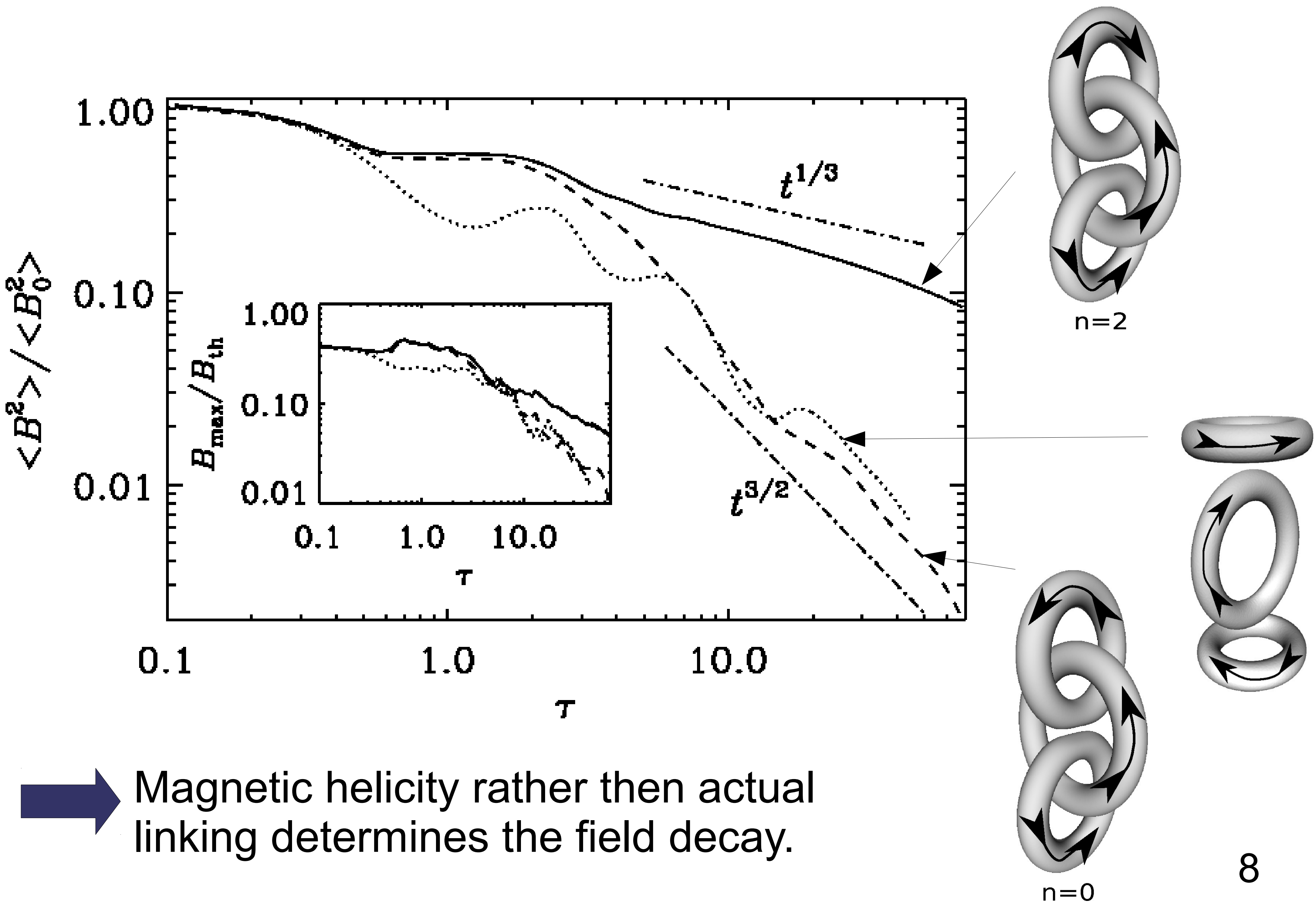


$H_{\mathrm{M}} = 0$



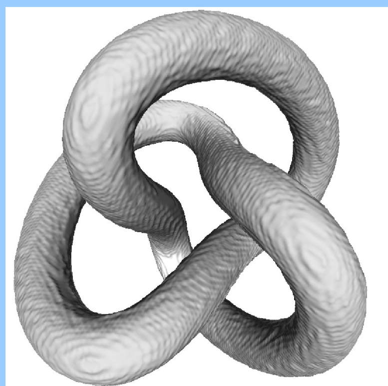
$H_{\mathrm{M}} \neq 0$

Interlocked flux rings

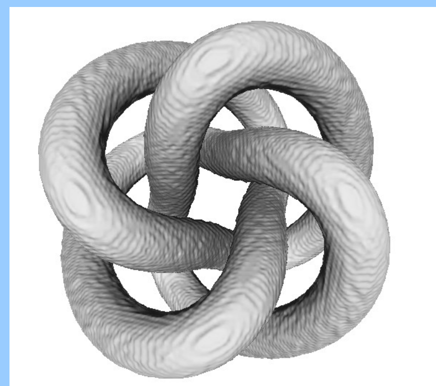


Magnetic helicity rather then actual linking determines the field decay.

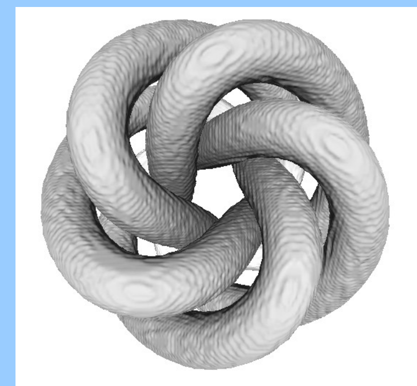
N-foil knots



3-foil



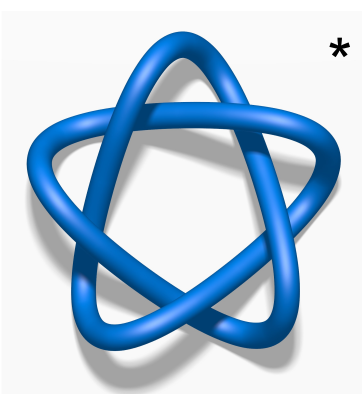
4-foil



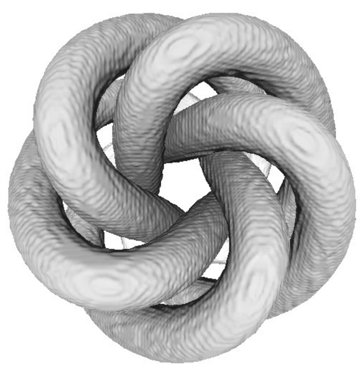
5-foil

6-foil

7-foil



$\neq$

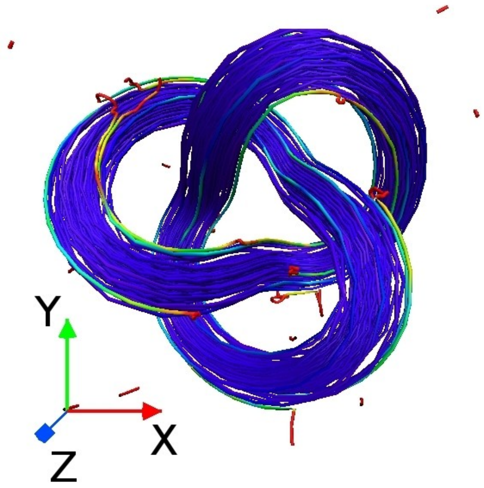


$$x(s) = \begin{pmatrix} (C + \sin s n_{\mathrm{f}}) \sin[s(n_{\mathrm{f}} - 1)] \\ (C + \sin s n_{\mathrm{f}}) \cos[s(n_{\mathrm{f}} - 1)] \\ D \cos s n_{\mathrm{f}} \end{pmatrix}$$

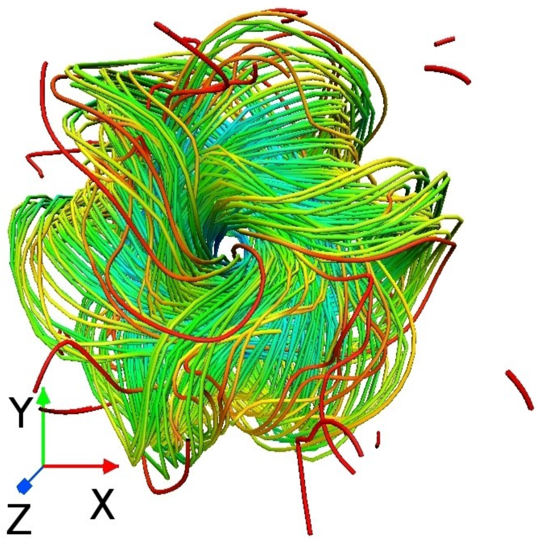
cinquefoil knot

* from Wikipedia, author: Jim.belk

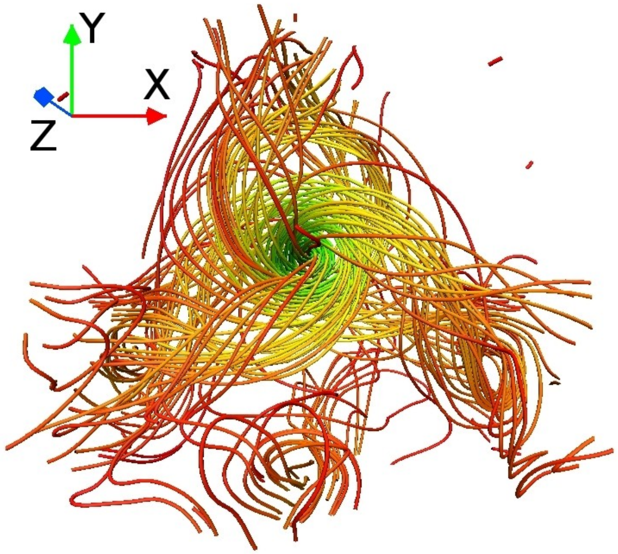
N-foil knots



$t = 0$



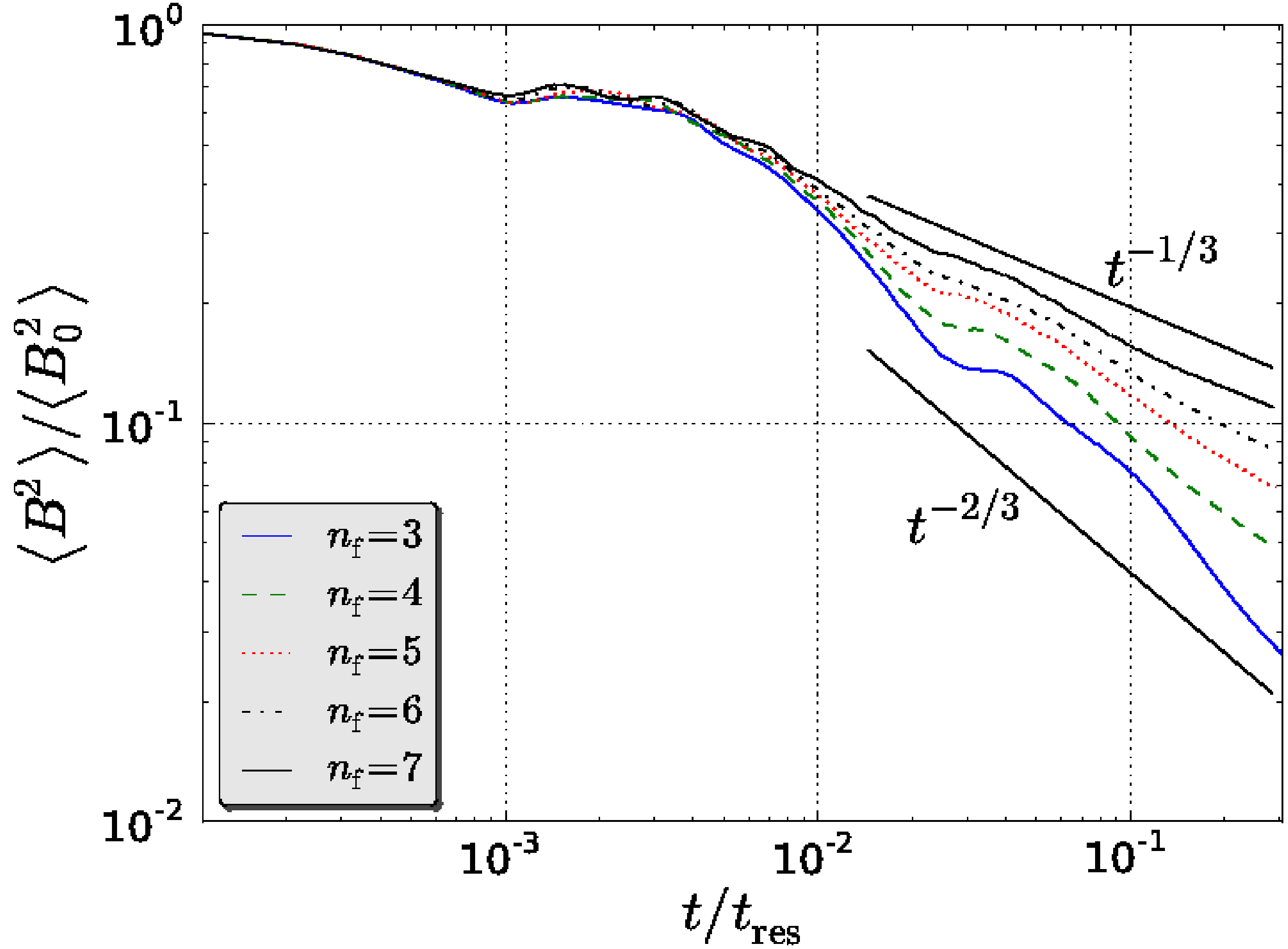
$t = 6$



$t = 39$

- Magnetic helicity is approximately conserved.
- Self-linking is transformed into twisting after reconnection.

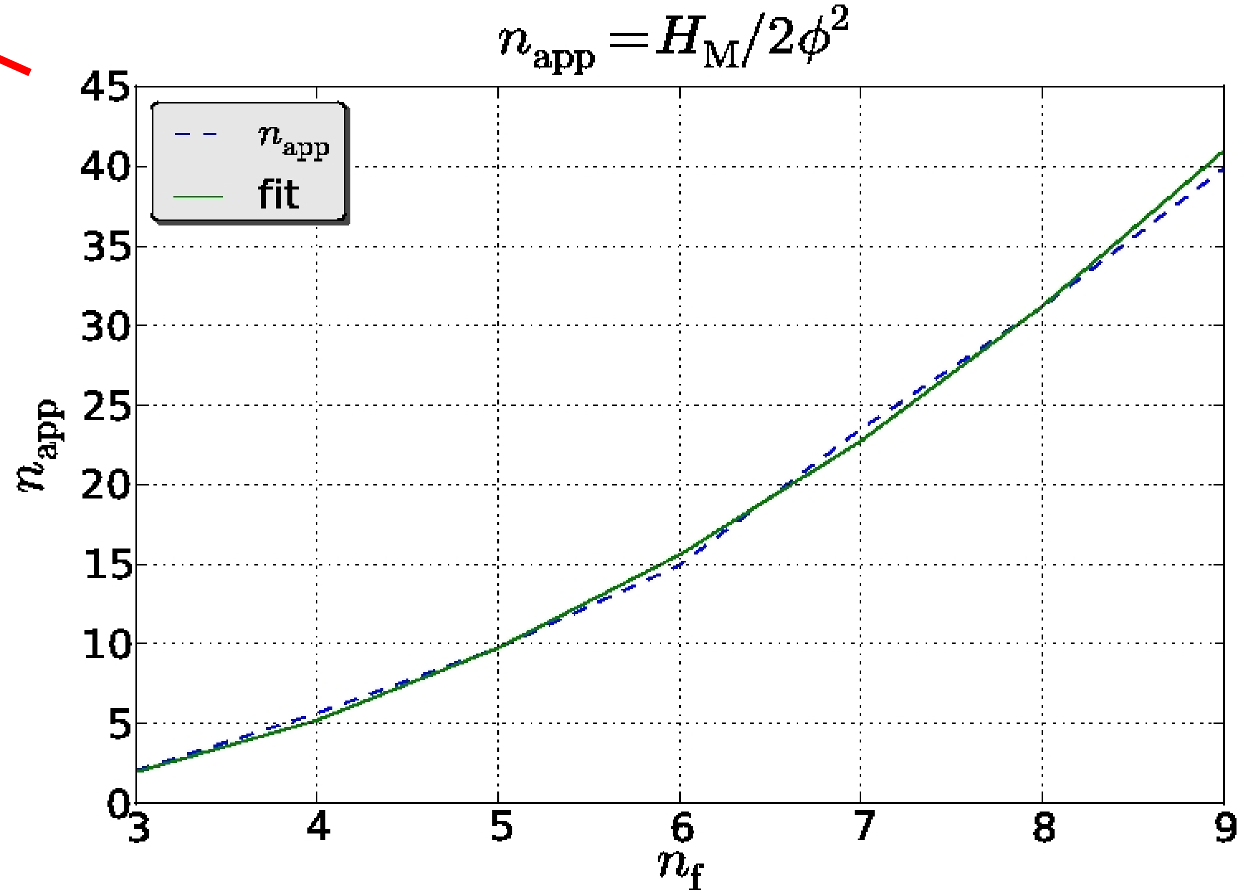
N-foil knots



Slower decay for higher $n_{\rm f}$.

N-foil knots

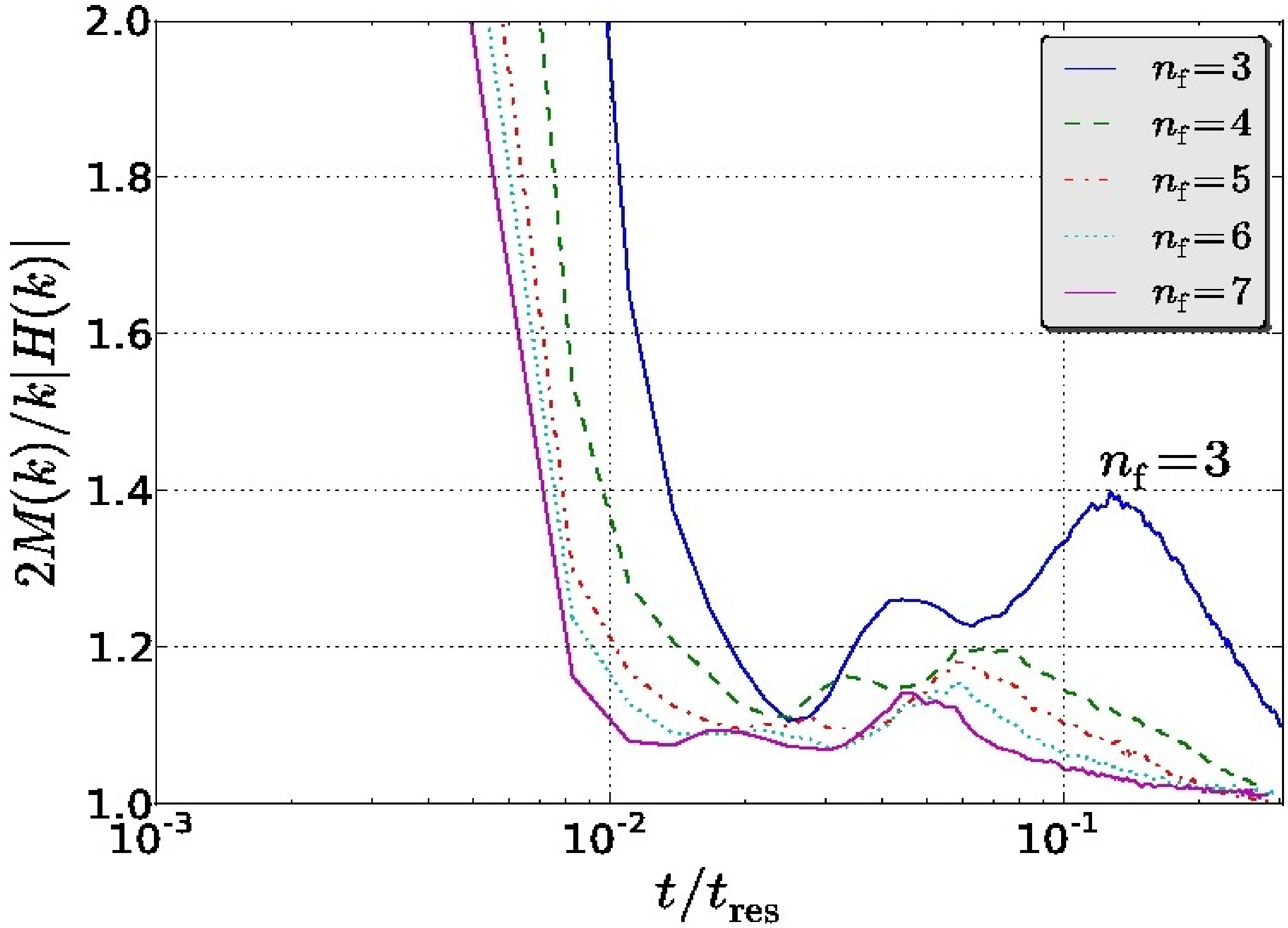
~~$H_\mathrm{M} = 2n\phi_1\phi_2$~~



$$H_\mathrm{M} = (n_\mathrm{f} - 2)n_\mathrm{f}\phi^2/2$$

N-foil knots

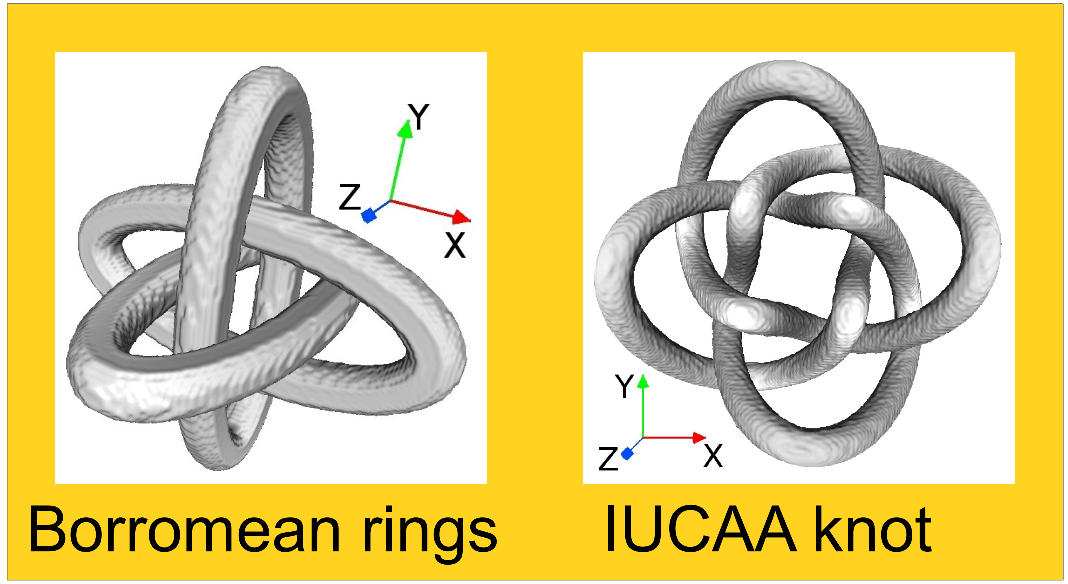
$2M(k)/(|H(k)|k)$



$k = 2$

Realizability condition more important for high $n_{\rm f}$.

IUCAA knot and Borromean rings

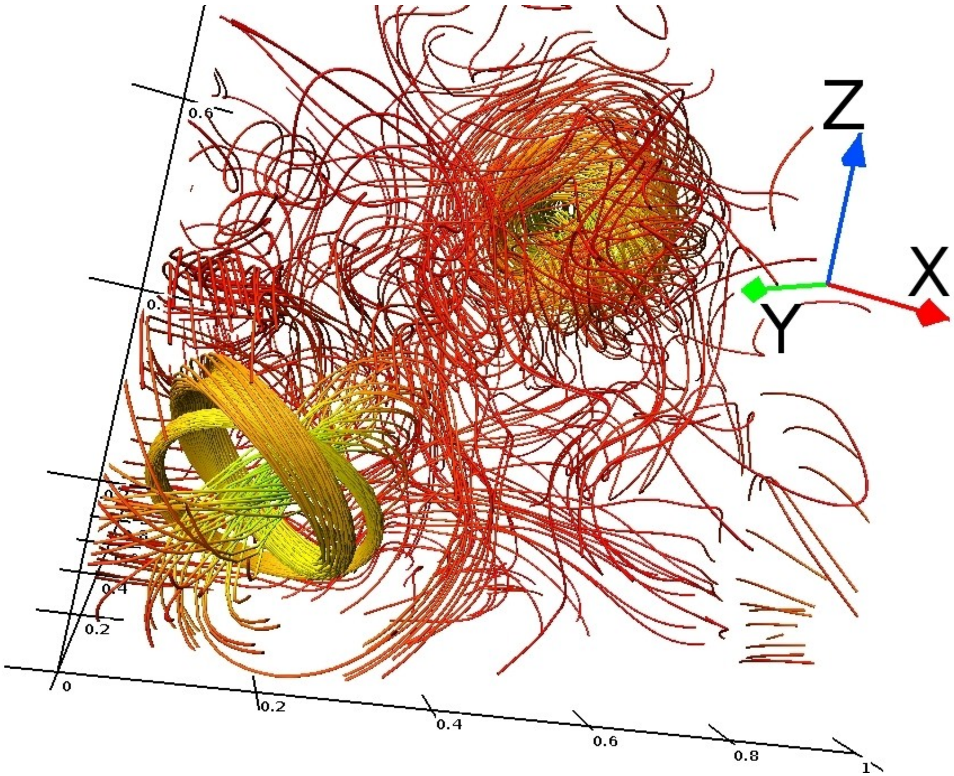


$$H_{\mathrm{M}} = 0$$

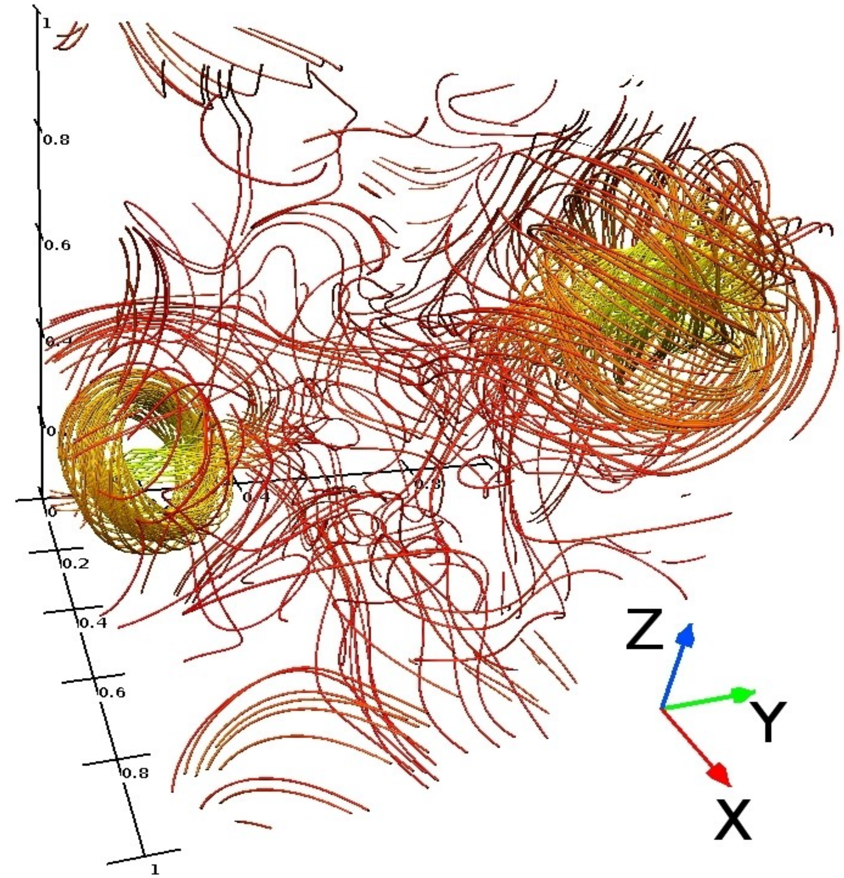
- Is magnetic helicity sufficient?
- Higher order invariants?

IUCAA = The Inter-University Centre for Astronomy and Astrophysics, Pune, India

Reconnection characteristics



$t = 70$

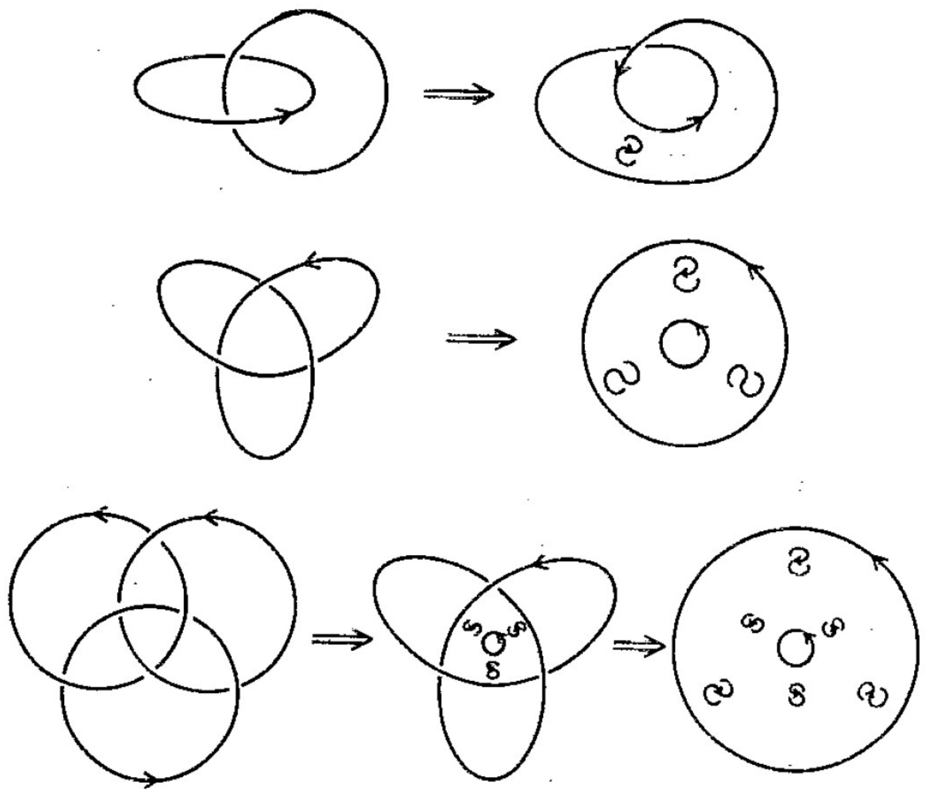


$t = 78$

3 rings → Twisted ring + interlocked rings → 2 twisted rings

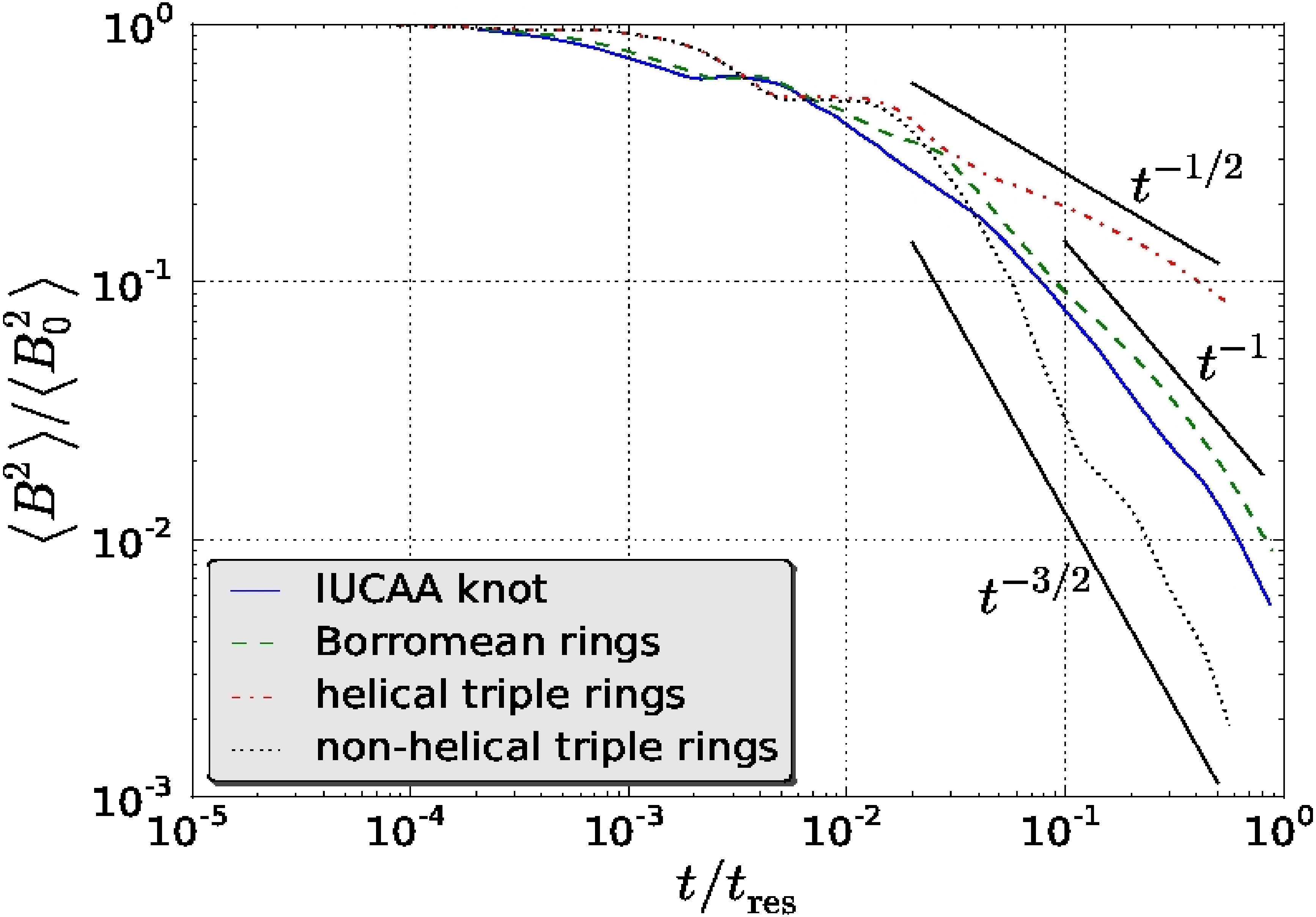
Reconnection characteristics

Conversion of linking into twisting

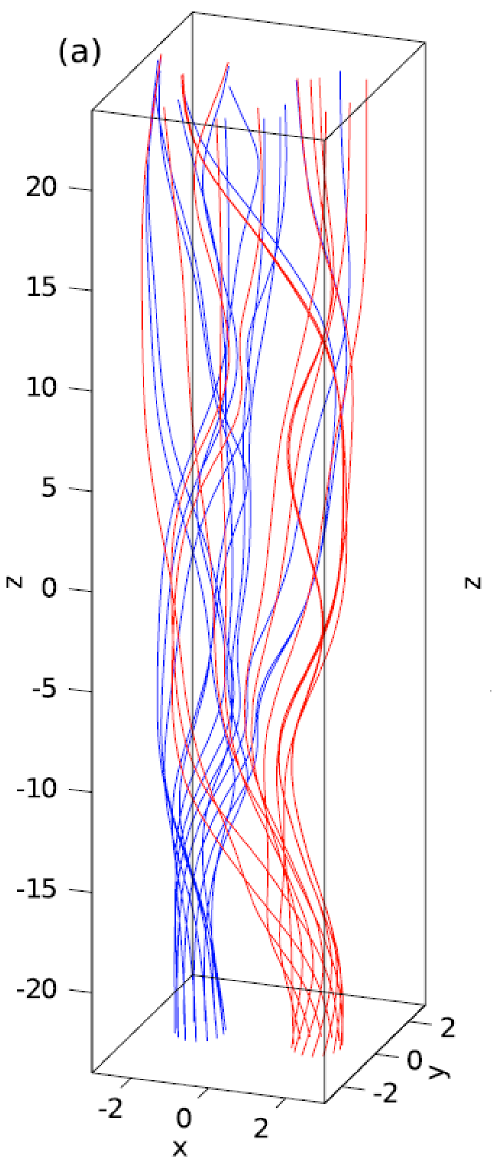


Ruzmaikin and Akhmetiev (1994)

Magnetic energy decay



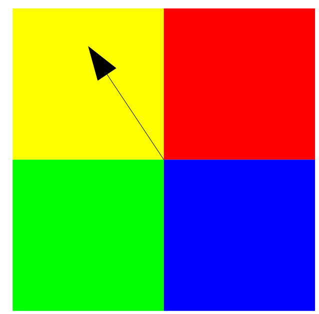
Fixed point index



mapping: $(x, y) \rightarrow \mathbf{F}_z(x, y)$

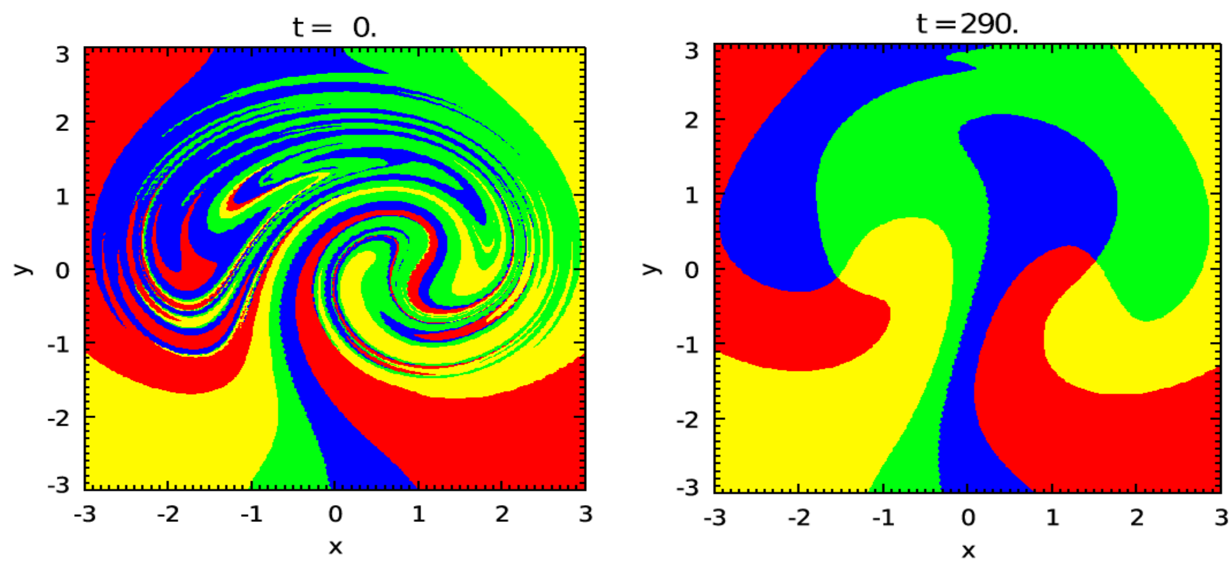
Fixed points: $\mathbf{F}_1(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$

Color coding:



Fixed point index:

$$T = \sum_i t_i \qquad t_i = \pm 1$$

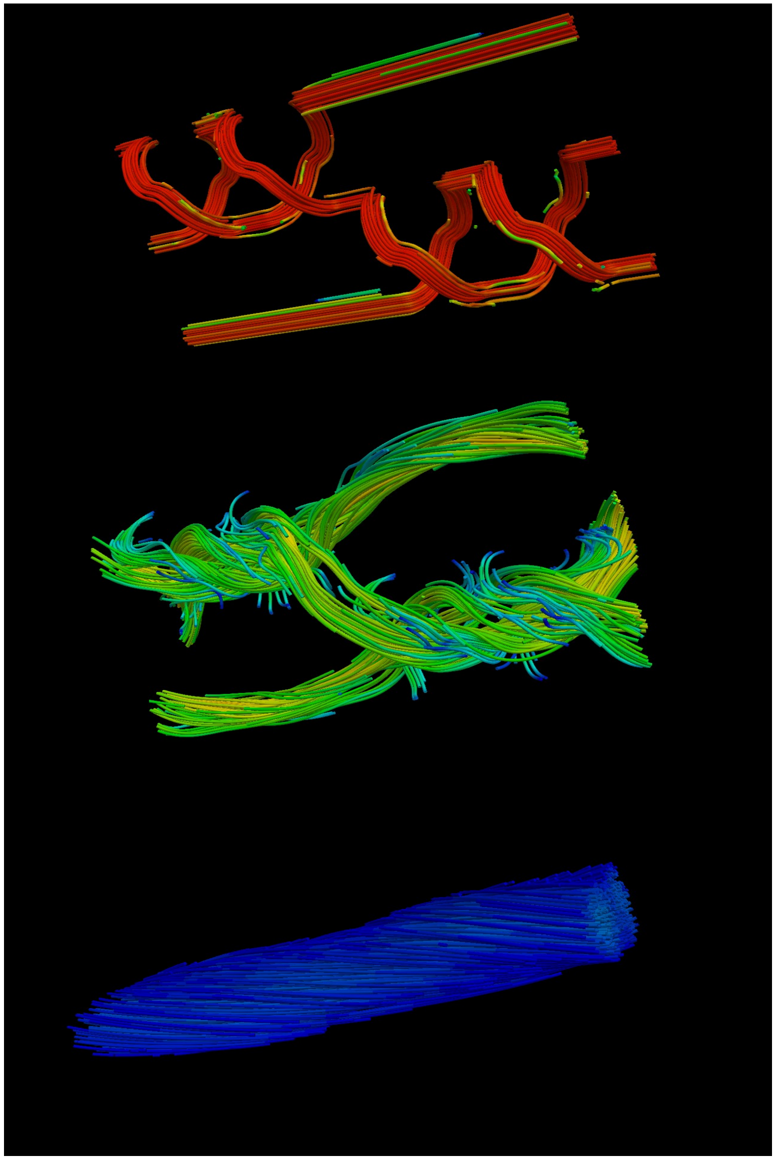


Yeates et al. 2011

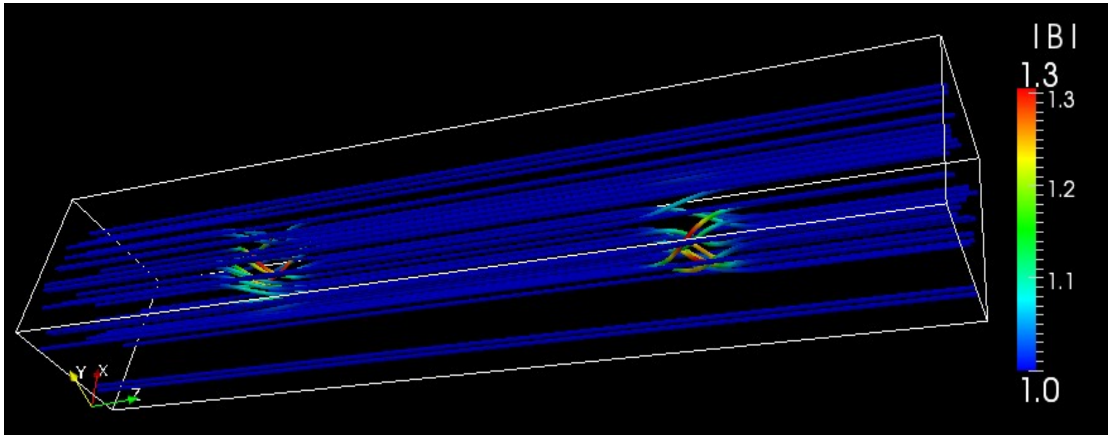
Magnetic braid configurations

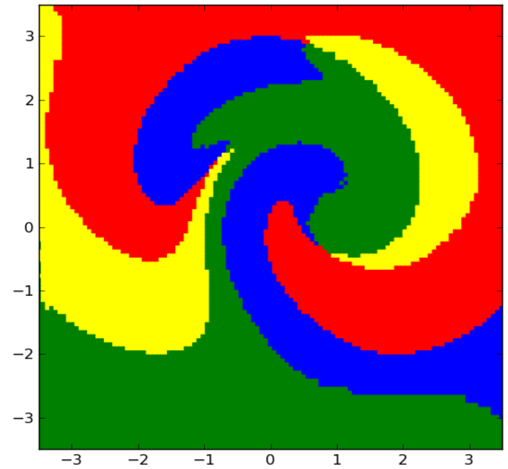
AAA (trefoil knot)

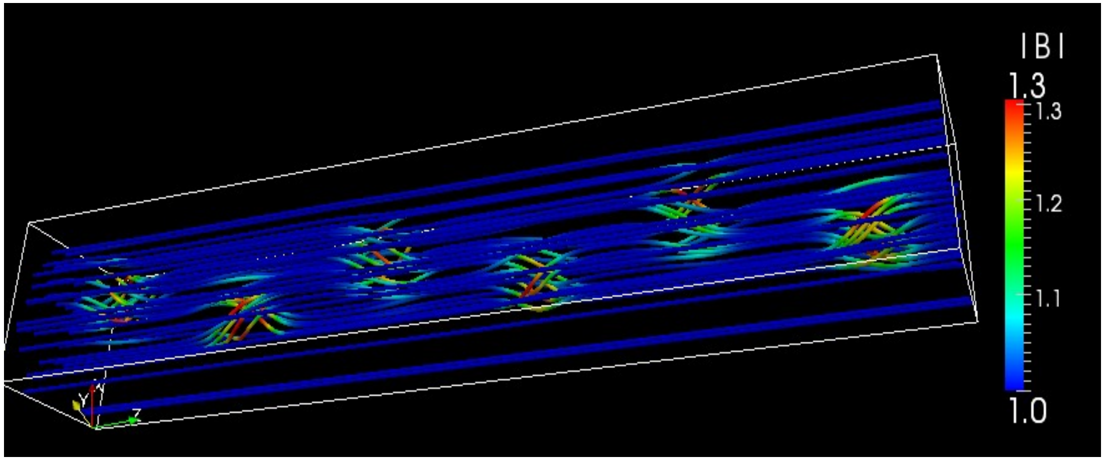
AABB (Borromean rings)

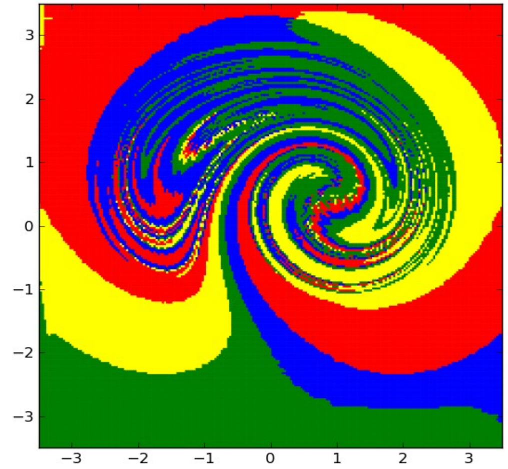


Field line tracing









Generalized flux function:

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

Reconnection rate:

$$\sum_i \frac{\mathrm{d}\mathcal{A}(\mathbf{x}_i)}{\mathrm{d}t}$$

Conclusions

- Topology *can* constrain field decay.
- Stronger packing for high n_{f} leads to different decay slopes.
- Higher order invariants?
- Isolated helical structures inhibit energy decay.
- Reconsider realizability condition.

- Apply fixed point method to knots (braids).
- Monitor the reconnection rate.

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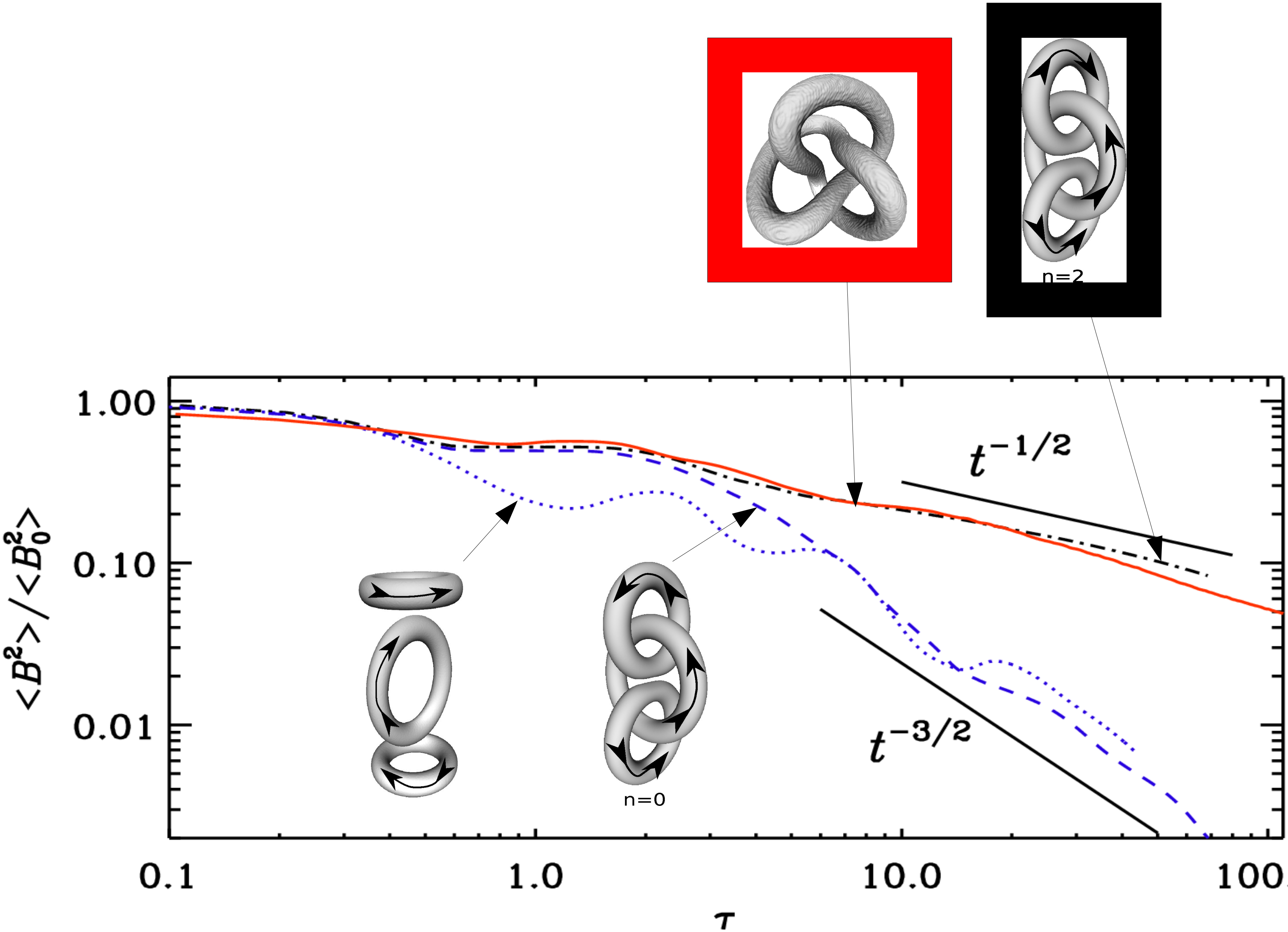
Magnetic energy decay



Simulations

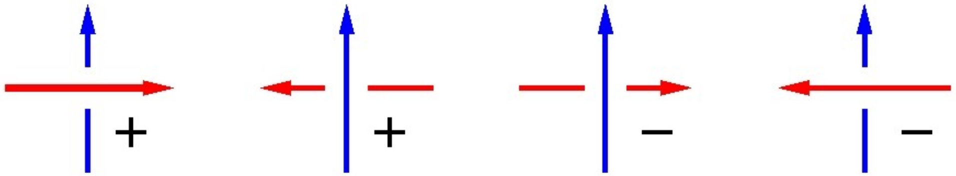
- 256^3 mesh point
- Isothermal compressible gas
- Viscous medium
- Periodic boundaries

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \eta \nabla^2 \mathbf{A}$$

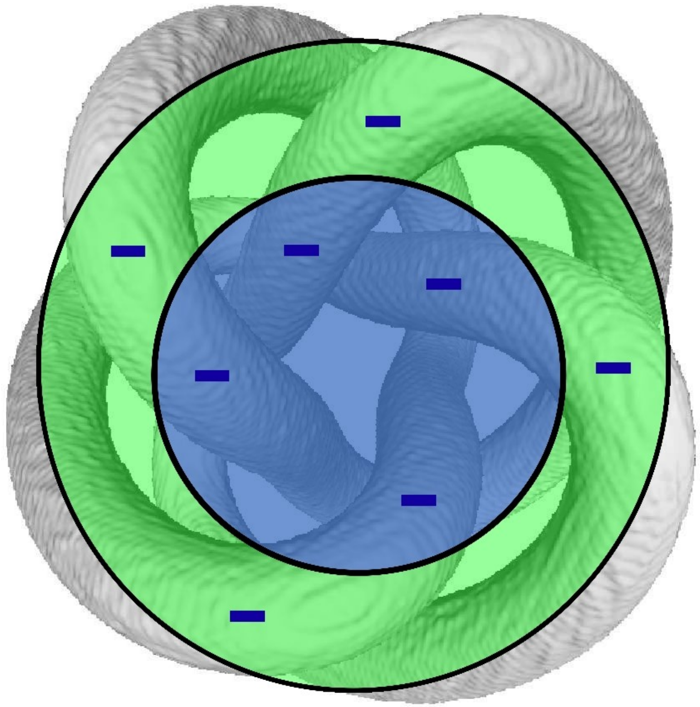
$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \boldsymbol{\nabla} \ln \rho + \mathbf{J} \times \mathbf{B}/\rho + \mathbf{F}_{\mathrm{visc}}$$

$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{U}$$

Linking number



$$n_{\mathrm{linking}} = (n_+ - n_-)/2$$



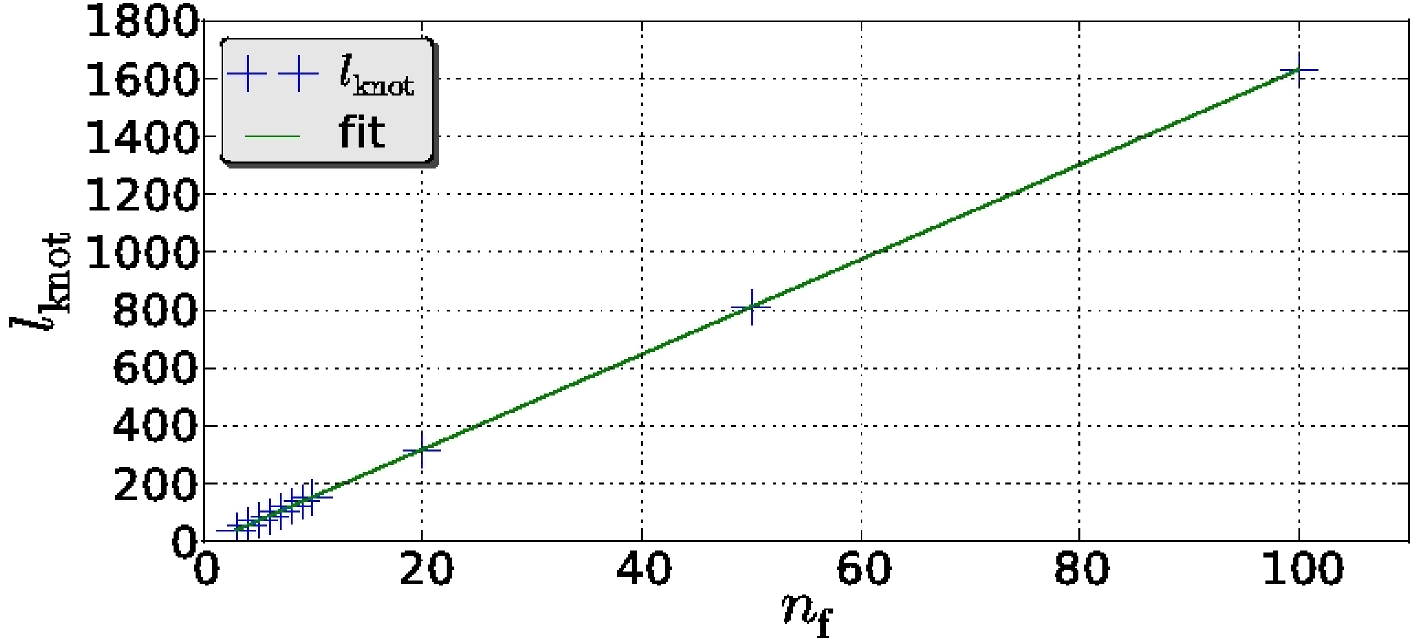
Sign of the crossings
for the 4-foil knot

Number of crossings increases like n_{f}^2

$$H_{\mathrm{M}} \propto n_{\mathrm{linking}}$$

$$H_{\mathrm{M}} \propto n_{\mathrm{f}}^2$$

Helicity vs. energy



$$E_{\mathrm{M}} \propto l_{\mathrm{knot}} \propto n_{\mathrm{f}}$$

$$H_{\mathrm{M}} \propto n_{\mathrm{f}}^2$$

Knot is more strongly packed with increasing n_{f}.

Magnetic energy is closer to its lower limit for high n_{f}.