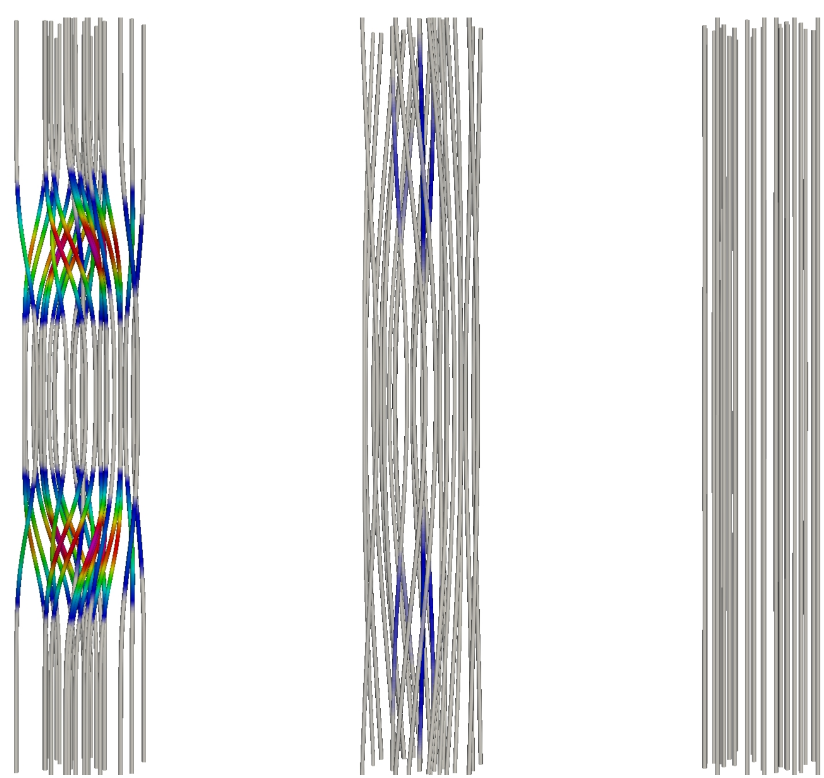
Ideal magnetic field relaxation and mimetic numerical operators

Simon Candelaresi



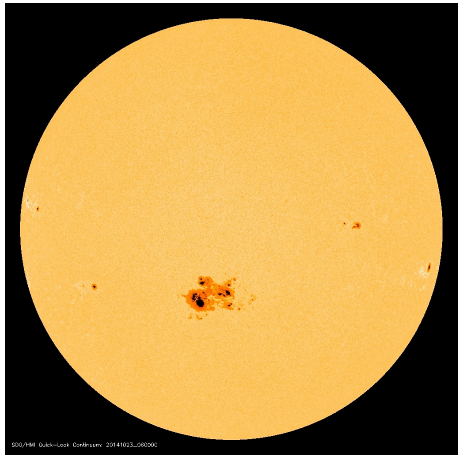
simon.candelaresi@gmail.com

Overview

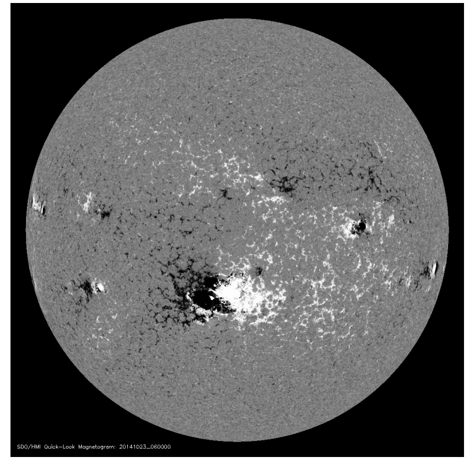
1. Physical Motivation
2. Magnetic Field Topology and Evolution
3. Force-Free Relaxation
4. Mimetic Methods
5. Latest Developments on Field Relaxation

Magnetic Fields in the Universe

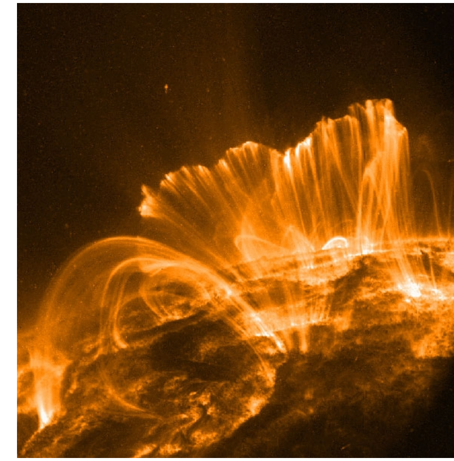
Sun:
2-2,000G



Continuum
2014-10-23 *(NASA)*

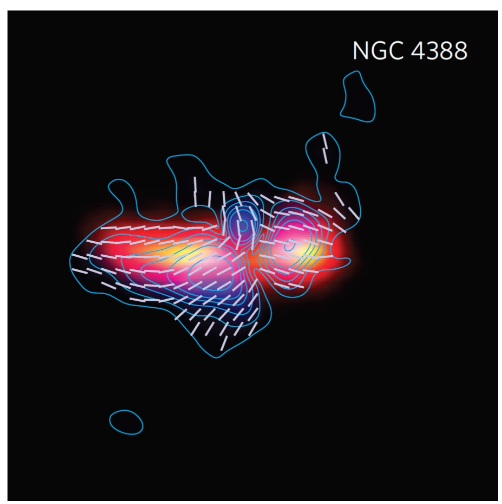


HMI (magnetic field)
2014-10-23 *(NASA)*



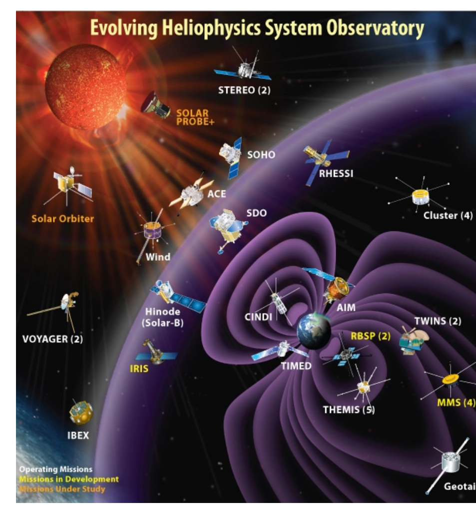
Coronal Loops *(NASA)*

Galaxies:
10e-6 G



Pfrommer (2010)
DOI: 10.1038/NPHYS1657

Earth:
0.1-1G



NASA

Field's Environment

Magnetically dominated:

magnetic pressure >> thermal pressure

$$B^2/(2\mu_0) \gg n k_{\rm B} T$$

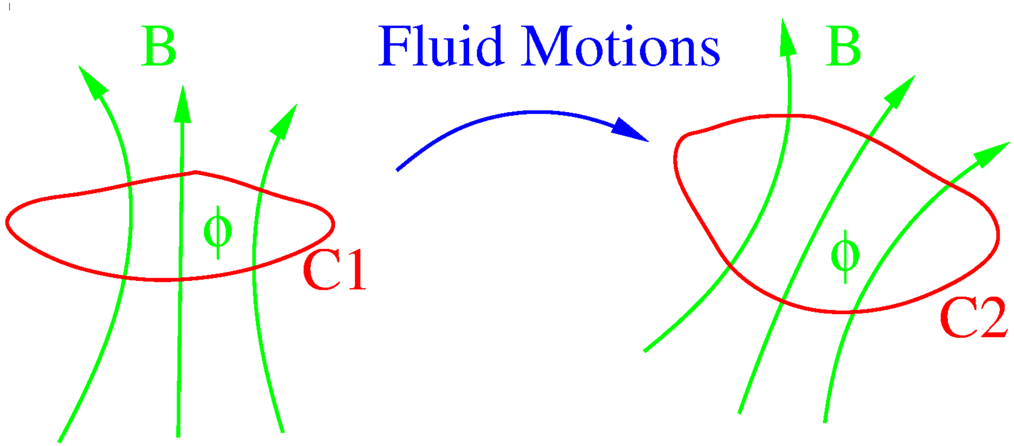
$$\beta = 2\mu_0 \frac{n k_{\rm B} T}{B^2} \ll 1$$

Solar corona: $\beta \approx 0.01$

Frozen-in magnetic flux:

magnetic resistivity small: $t_{\rm dissipation} \gg t_{\rm dynamical}$

⇨ Magnetic field is *frozen-in* to the fluid.



Batchelor (1950)
DOI: 10.1098/rspa.1950.0069

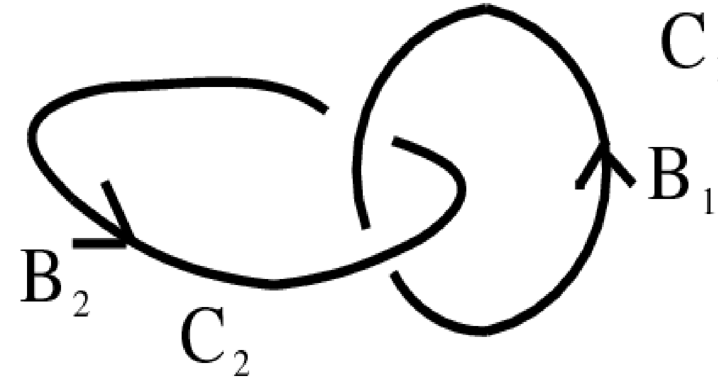
Magnetic Field Topology

Measure for the topology:

$$H_{\mathrm{M}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 2n\phi_1\phi_2$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B} \qquad \phi_i = \int_{S_i} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$

$n =$ number of mutual linking



Moffatt (1969)
DOI: 10.1017/S0022112069000991

Conservation of magnetic helicity:

$$\lim_{\eta \to 0} \frac{\partial}{\partial t} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = 0 \qquad \eta = \text{magnetic resistivity}$$

Realizability condition:

$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

Arnold (1974)

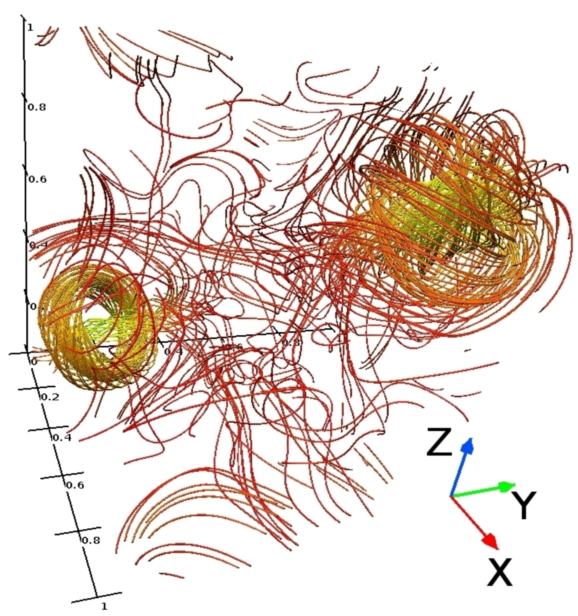
Magnetic energy is bound from below by magnetic helicity.

Relaxation of Magnetic Fields

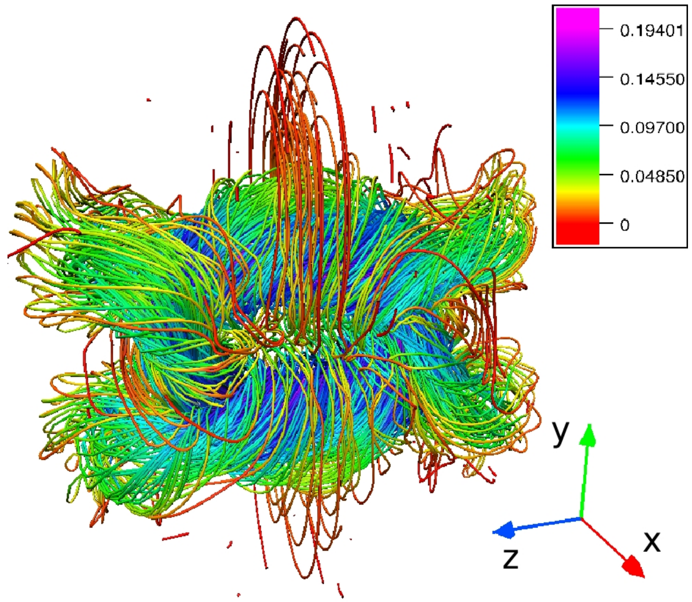
Global magnetic helicity conservation

⇨ final state is linearly force-free: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

Woltjer (1958)
DOI: 10.1073/pnas.44.9.833



Candelaresi (2011)
DOI: 10.1103/PhysRevE.84.016406



Del Sordo (2010)
DOI: 10.1103/PhysRevE.81.036401

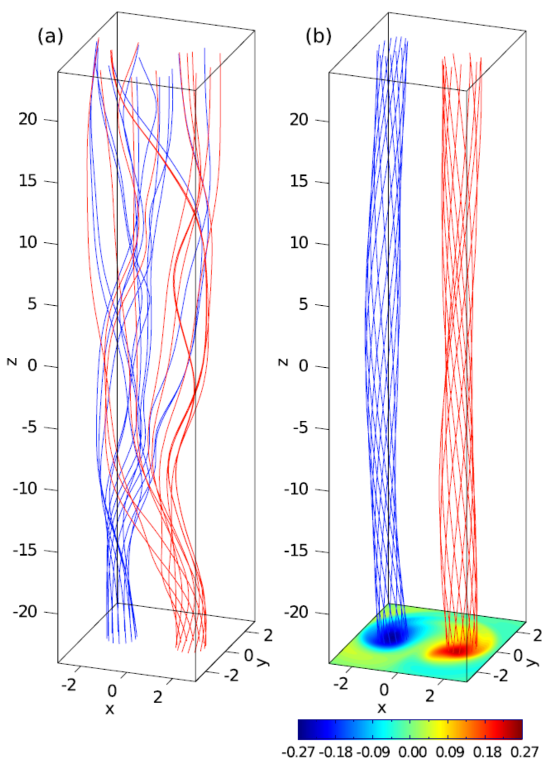
Taylor Relaxation

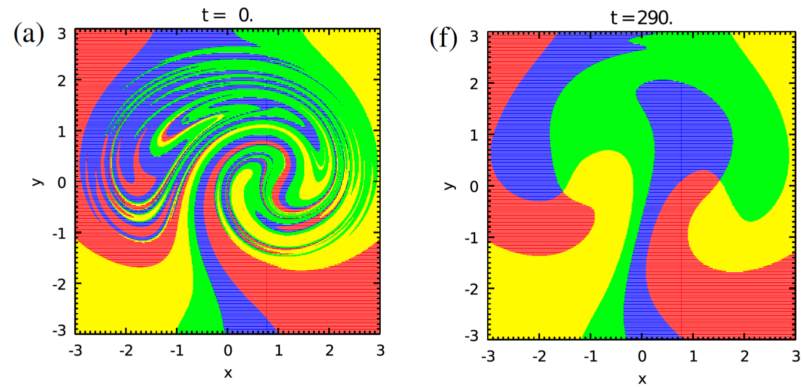
Field line magnetic helicity conservation

final state is non-linear force-free: $\nabla \times \mathbf{B} = \lambda(a, b)\mathbf{B}$

Taylor (1974)
DOI: 10.1103/PhysRevLett.33.1139

Does the system always reach this state?



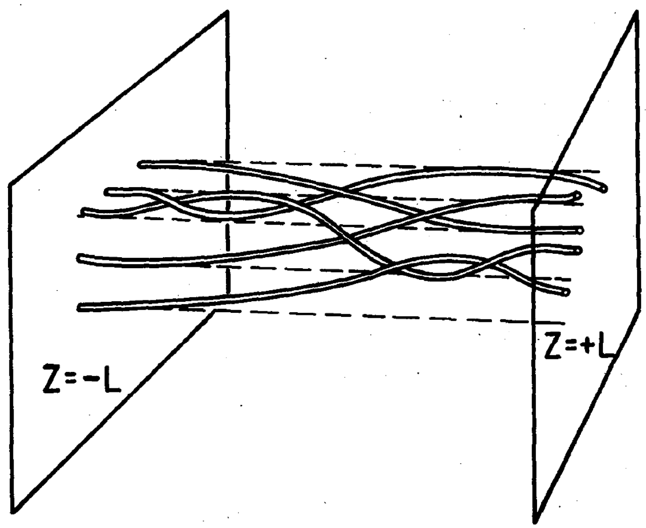


Yeates (2010)
DOI: 10.1103/PhysRevLett.105.085002

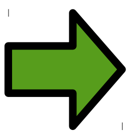
Not necessarily. Additional topological degree must be conserved.

Parker Conjecture

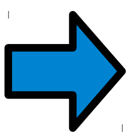
Line-tied magnetic field bound within two plates.



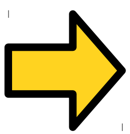
Parker (1972)
DOI: 10.1086/151512



Hydrostatic equilibrium only if field variation is uniform along the field.



If equilibrium is not reached field lines merge and magnetic energy is dissipated (topological dissipation).



For complex topologies formation of singular current sheets.

Force-Free Magnetic Fields

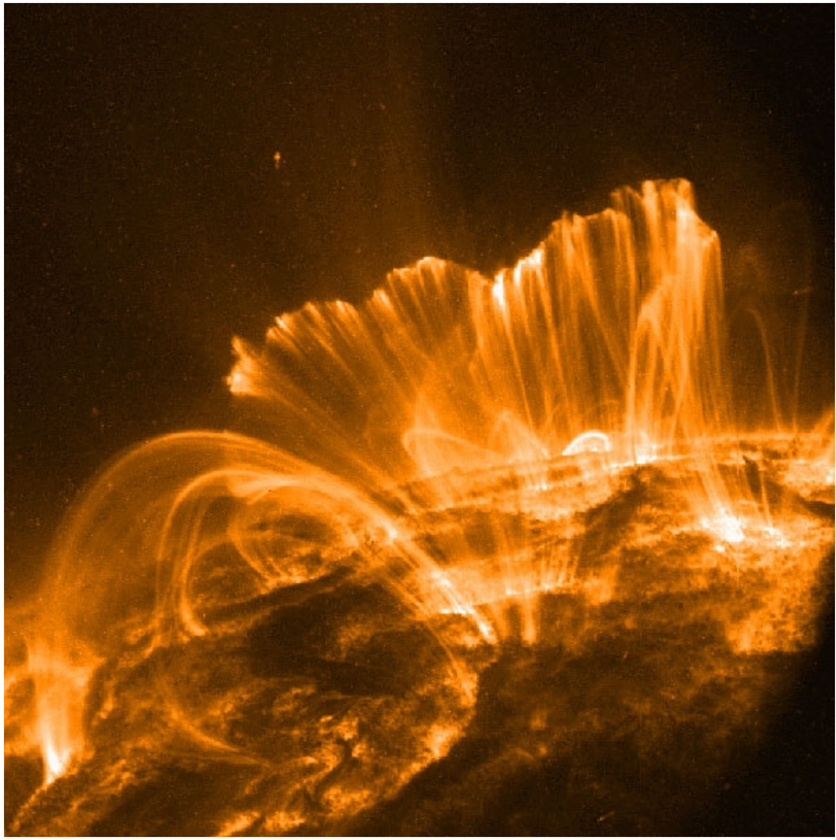
Solar corona: low plasma beta and magnetic resistivity

- ⇒ Force-free magnetic fields
- ⇒ Minimum energy state

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \;\Leftrightarrow\; \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

Beltrami field



NASA

Problem:
Find a force-free state for a magnetic field with given topology.
Current sheets?

Here:
Numerical method for finding such states.

Ideal Field Relaxation

Ideal induction eq.: $$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

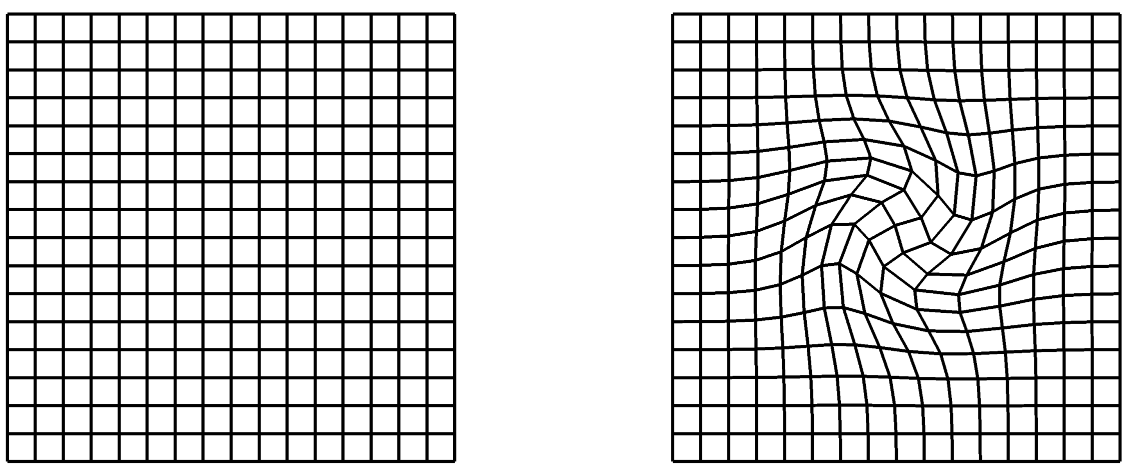
Frozen in magnetic field.

(Batchelor, 1950)

But: Numerical diffusion in finite difference Eulerian codes.

Solution: Lagrangian description of moving fluid particles:

$\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$ $\mathbf{x}(\mathbf{X}, t)$



Ideal Field Relaxation

Field evolution: $$B_i(\mathbf{X},t) = \frac{1}{\Delta}\sum_{j=1}^{3}\frac{\partial x_i}{\partial X_j}B_j(\mathbf{X},0)$$

$$\Delta = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

Preserves topology and divergence-freeness.

Grid evolution: $$\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \mathbf{u}\left(\mathbf{x}(\mathbf{X},t),t\right)$$

Magneto-frictional term: $\mathbf{u} = \gamma \mathbf{J}\times\mathbf{B} \qquad \mathbf{J} = \nabla\times\mathbf{B}$

$$\Rightarrow \frac{\mathrm{d}E_{\mathrm{M}}}{\mathrm{d}t} < 0$$

(Craig and Sneyd 1986)

Numerical Curl Operator

Compute $\mathbf{J} = \nabla \times \mathbf{B}$ on a distorted grid:

$$\frac{\partial B_i}{\partial x_j} = X_{\alpha,j}(x_{i,\alpha\beta} B^0_\beta \Delta^{-1} + x_{i,\beta} B^0_{\beta,\alpha} \Delta^{-1} - x_{i,\beta} B^0_\beta \Delta^{-2} \Delta_{,\alpha})$$

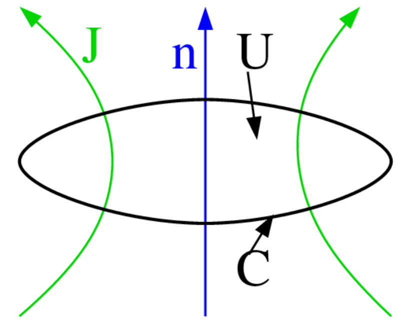
$$B^0_i = B_i(0)$$

(Craig and Sneyd 1986)

- Multiplication of several terms leads to high numerical errors.
- Current not divergence free: $\nabla \cdot \mathbf{J} \neq 0$
- Only reaching a certain force-freeness. *(Pontin et al. 2009)*

Mimetic Numerical Operators

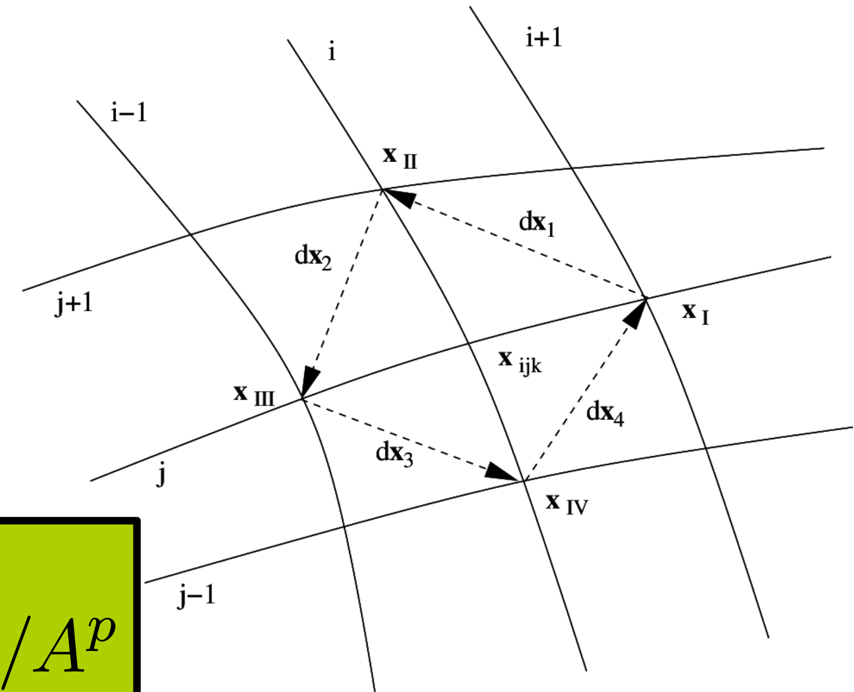
$$I = \int_U \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}S = \oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r}$$



Discretized:

$$I \approx \mathbf{J}(\mathbf{X}_{ijk})\cdot\mathbf{n}A = \sum_{r=1}^{4}\mathbf{B}_r\cdot\mathrm{d}\mathbf{x}_r$$

$$\mathbf{J}(\mathbf{X}_U) \approx \mathbf{J}(\mathbf{X}_{\mathbf{ijk}}), \quad \mathbf{X}_U \in U$$



3 planes will give 3 l.i. normal vectors:

$$I^p = \mathbf{J}(\mathbf{X}_{ijk})\cdot\mathbf{n}^p = \sum_{r=1}^{4}\mathbf{B}_r^p\cdot\mathrm{d}\mathbf{x}_r/A^p$$

Inversion yields $\mathbf{J}$ with $\nabla\cdot\mathbf{J} = 0$.

(Hyman, Shashkov 1997)

$$\nabla_\mathrm{M}\times\nabla_\mathrm{M}\phi = 0$$

$$\nabla_\mathrm{M}\cdot\nabla_\mathrm{M}\times\mathbf{A} = 0$$

Simulations

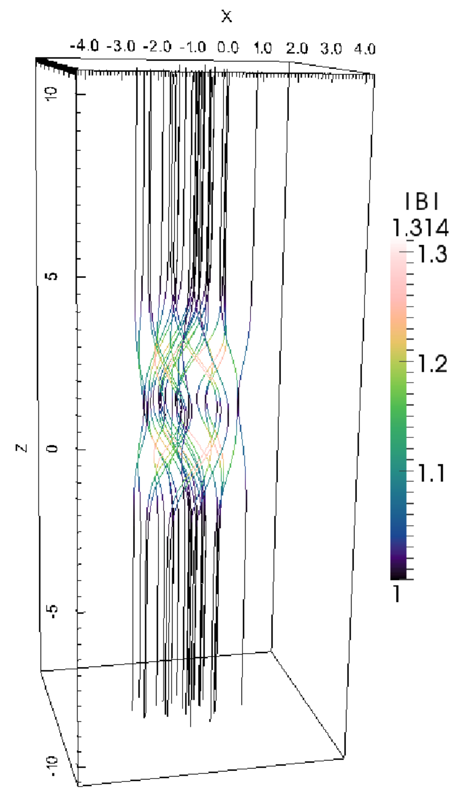
• GPU code GLEMuR (**G**pu-based **L**agrangian mim**E**tic **M**agnetic **R**elaxation)

• line tied boundaries

• mimetic vs. classic

(Candelaresi et al. 2014)

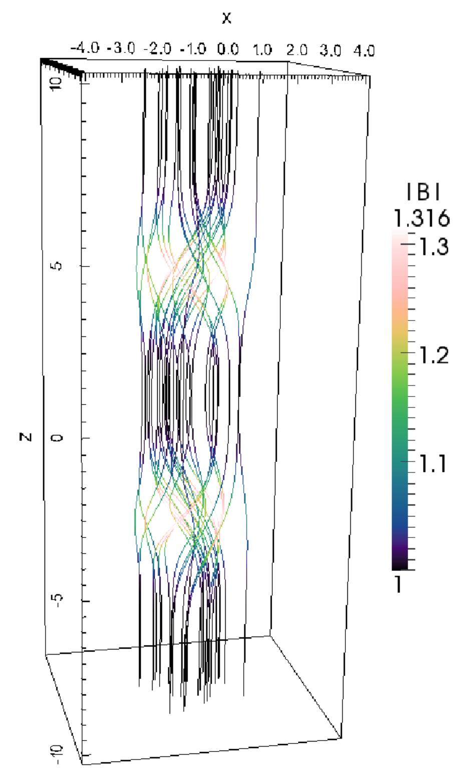
Nvidia Tesla K40



we know:

$$\lim_{t\to\infty} \mathbf{B}(t)$$

$$\lim_{t\to\infty} \mathbf{x}(t)$$



we know:

$$\lim_{t\to\infty} \mathbf{B}(t)$$

Quality Parameters

Deviation from the expected relaxed state:

$$\sigma_{\mathbf{x}} = \sqrt{\frac{1}{N}\sum_{ijk}(\mathbf{x}(\mathbf{X}_{ijk}) - \mathbf{x}_{\mathrm{relax}}(\mathbf{X}_{ijk}))^2}$$

$$\sigma_{\mathbf{B}} = \sqrt{\frac{1}{N}\sum_{ijk}(\mathbf{B}(\mathbf{X}_{ijk}) - \mathbf{B}_{\mathrm{relax}}(\mathbf{X}_{ijk}))^2}$$

Free magnetic energy:

$$E_{\mathrm{M}}^{\mathrm{free}} = E_{\mathrm{M}} - E_{\mathrm{M}}^{\mathrm{bkg}}$$

$$E_{\mathrm{M}} = \int_V \mathbf{B}^2/2 \; \mathrm{d}V \qquad \mathbf{B}^{\mathrm{bkg}} = B_0 \hat{e}_z$$

Quality Parameters

For a force-free field: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

➡ Force-free parameter does not change along field lines.

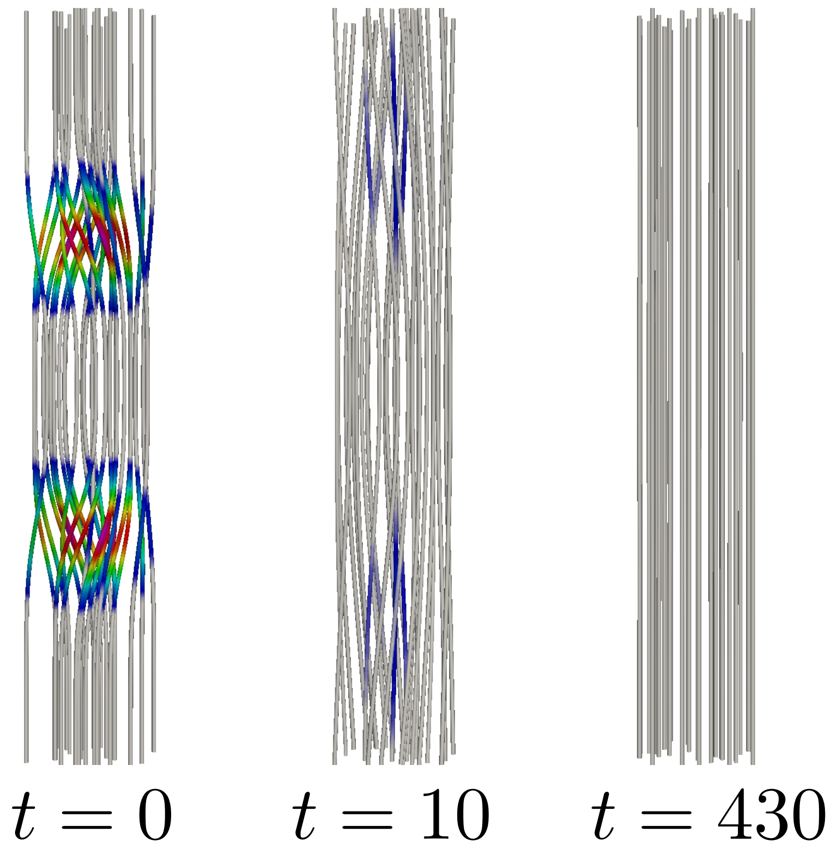
➡ Measure the change of $\alpha^* = \dfrac{\mathbf{J} \cdot \mathbf{B}}{\mathbf{B}^2}$ along field lines:

$$\epsilon^* = \max_{i,j} \left(a_r \frac{\alpha^*(\mathbf{X}_i) - \alpha^*(\mathbf{X}_j)}{|\mathbf{X}_i - \mathbf{X}_j|} \right); \quad \mathbf{X}_i, \mathbf{X}_j \in s_\alpha$$

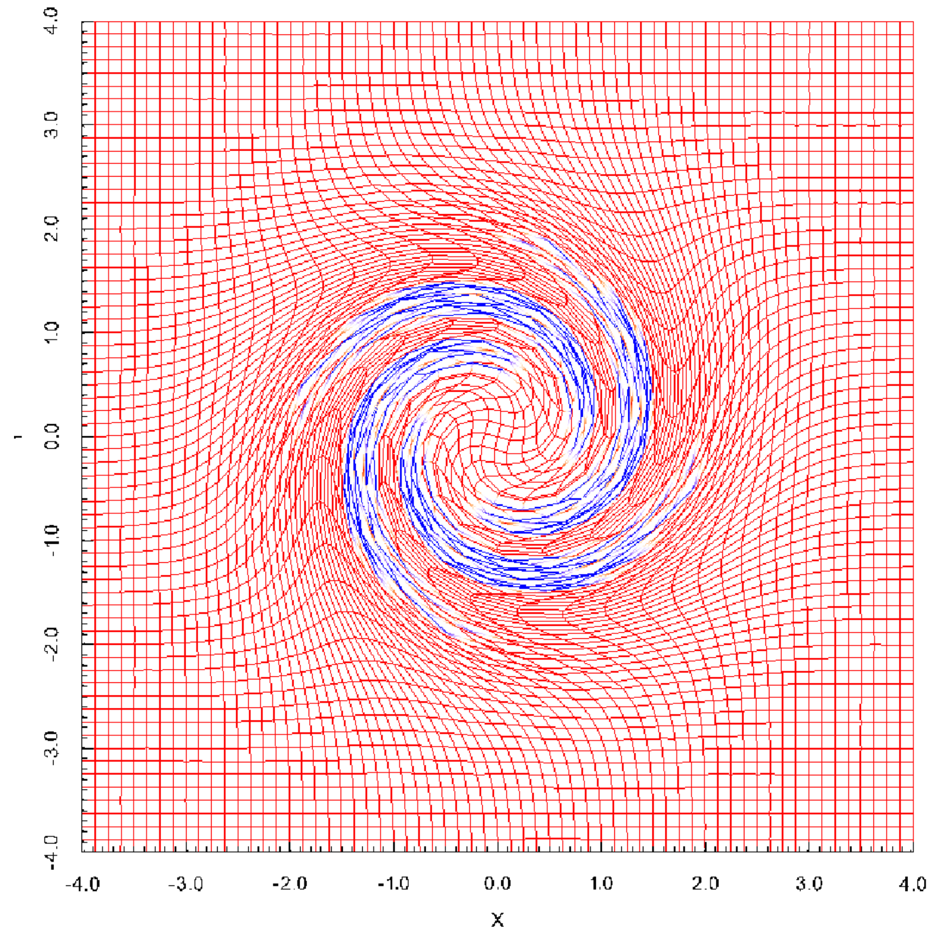
Particular field line: $s_\alpha = \{(0, 0, Z) : Z \in [-L_z/2, L_z/2]\}$

Field Relaxation

Magnetic streamlines:

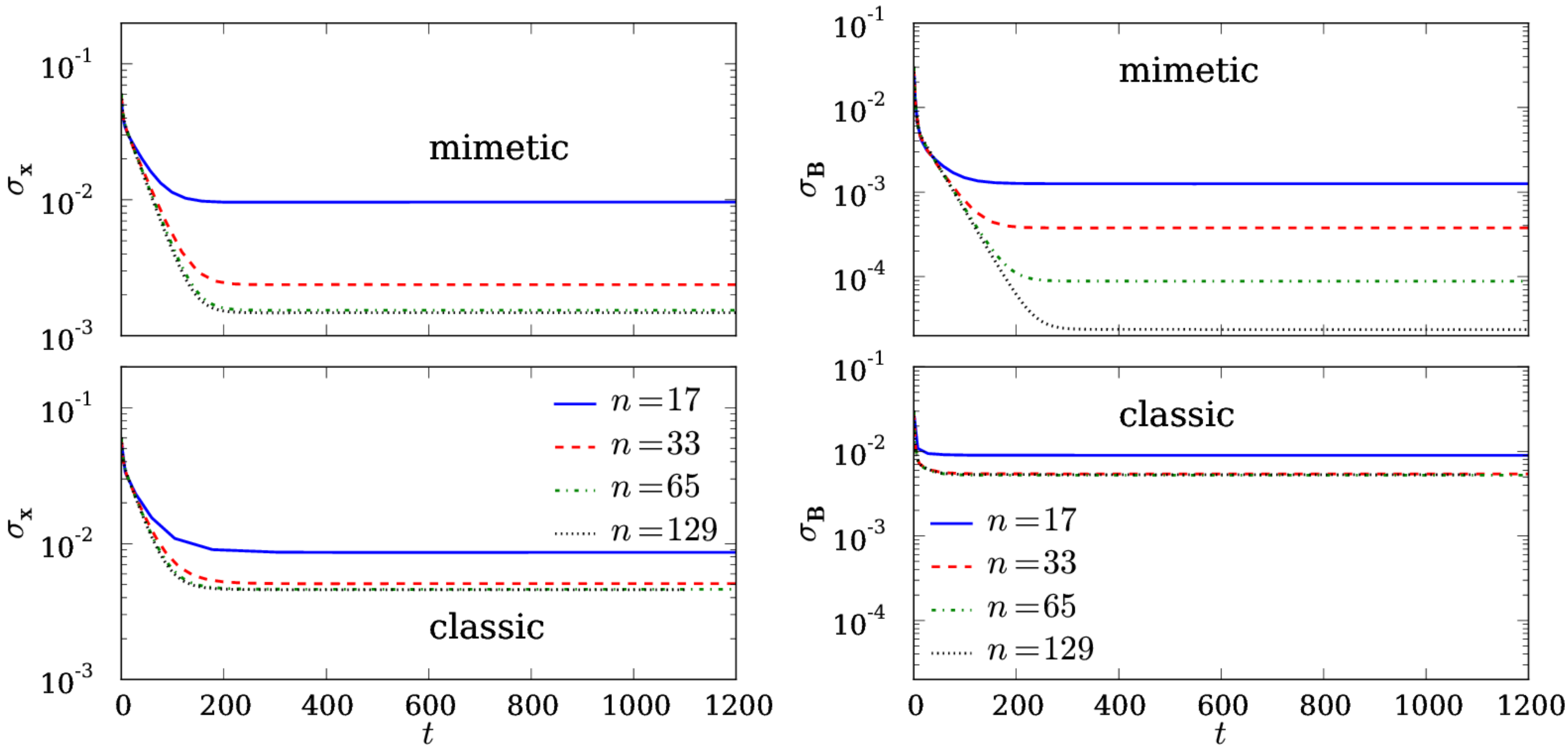


Grid distortion at mid-plane:



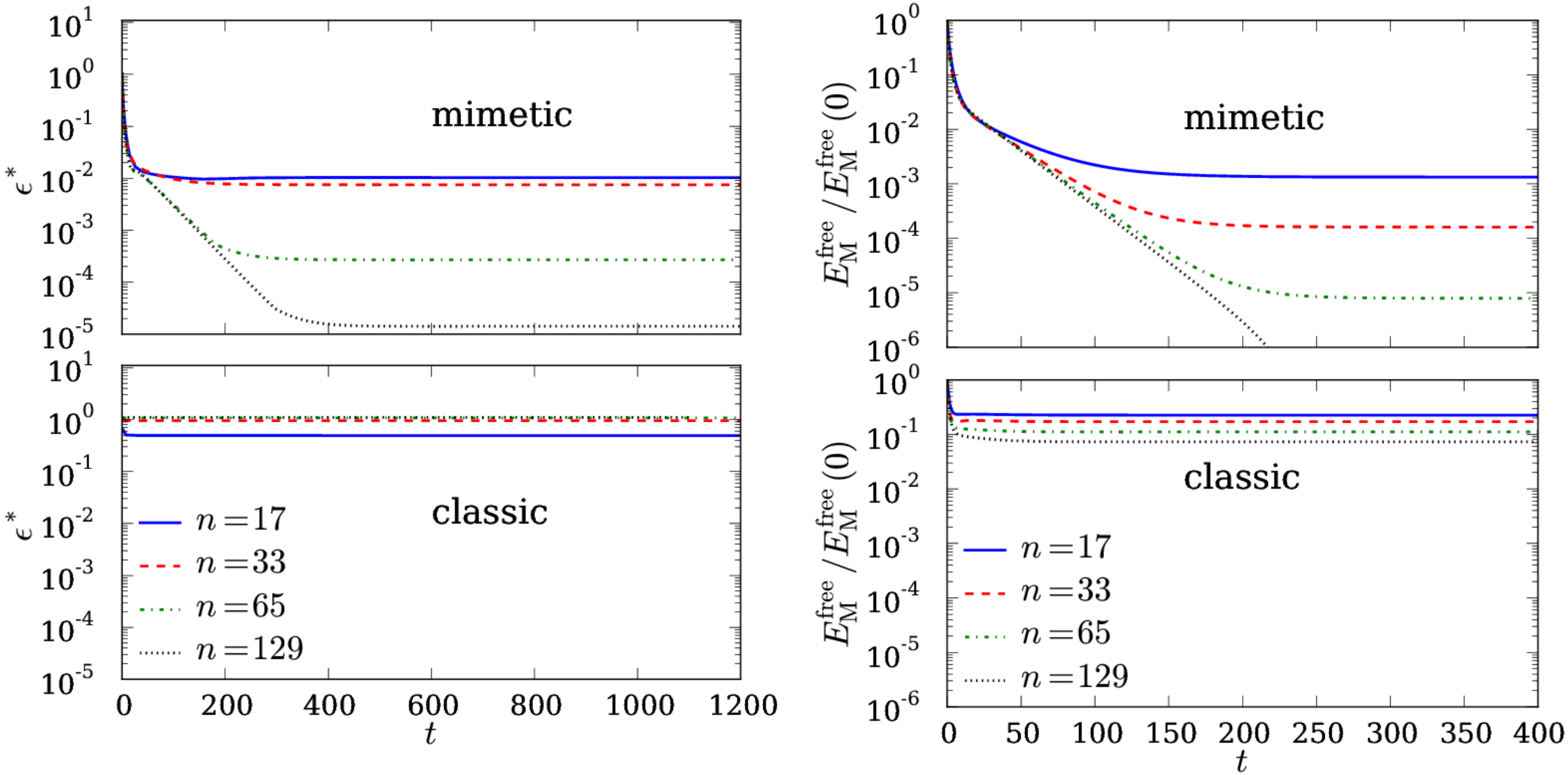
movie

Relaxation Quality



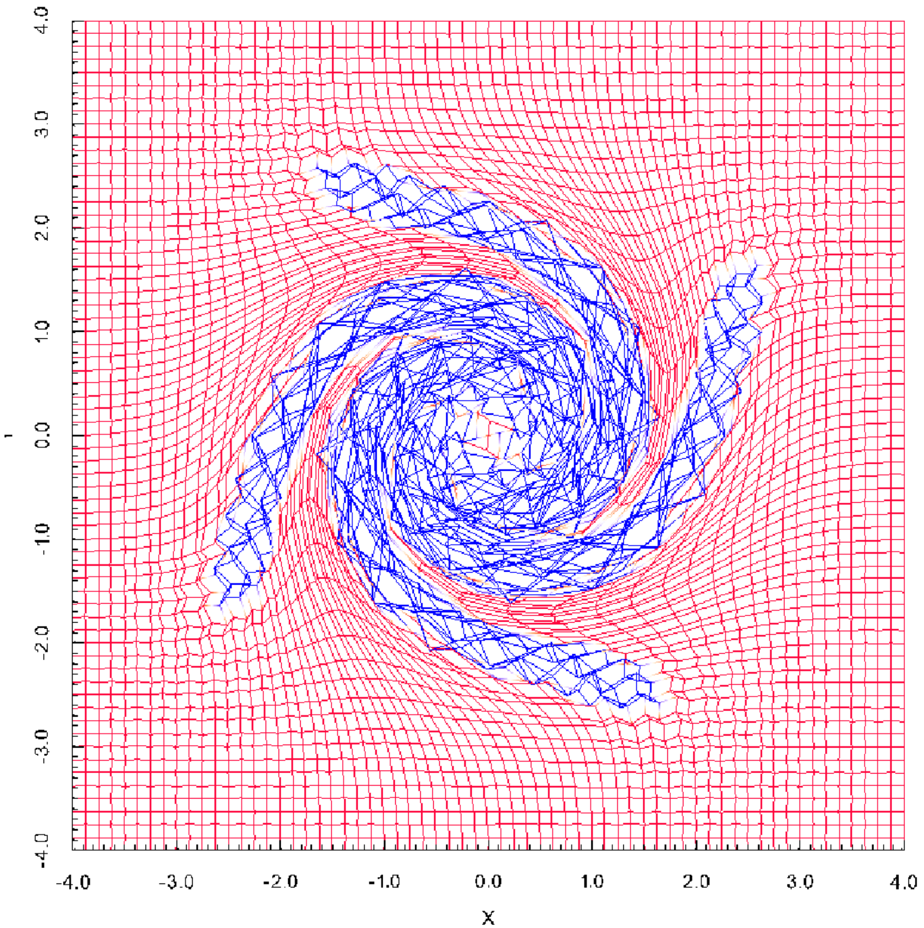
Closer to the analytical solution by 3 orders of magnitude.

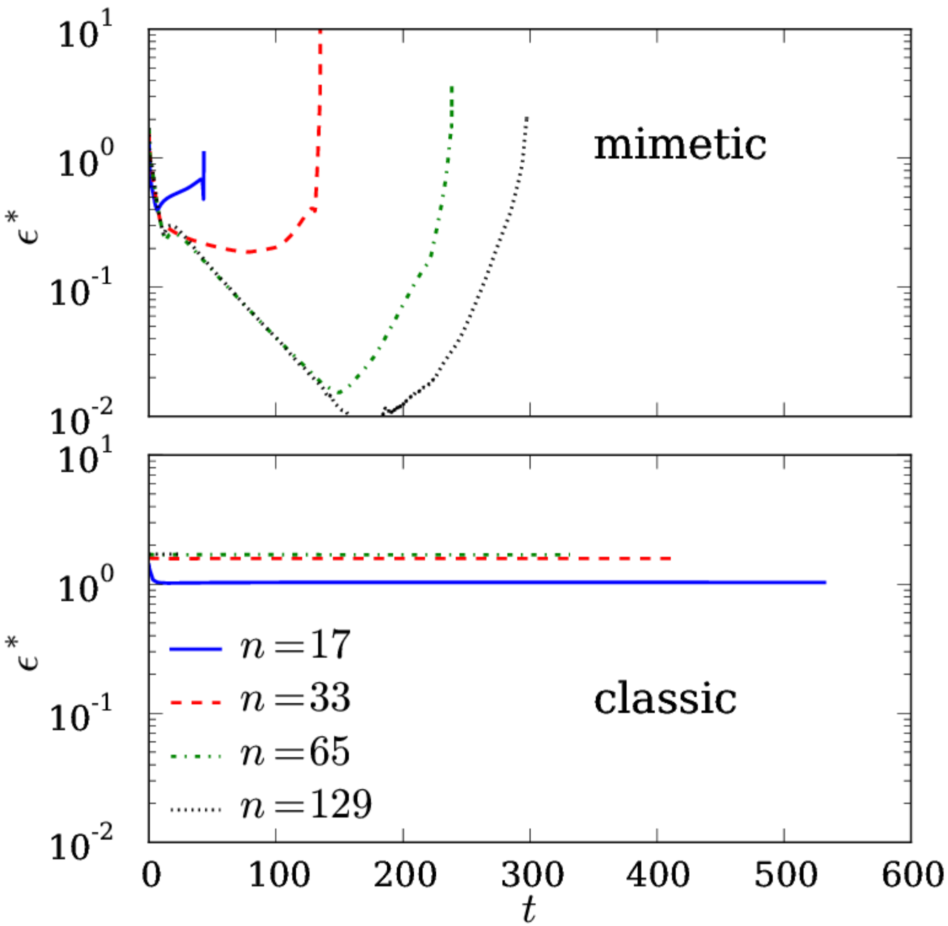
Relaxation Quality



Closer to force-free state by 5 orders of magnitude.

Limitations





red: convex
blue: concave

For concave cells the method becomes unstable.
But: results before crash better than classic method.

Code Details

- written in C++
- running in GPUs
- VTK data format
- 6th order Runge-Kutta time stepping
- periodic and line-tied boundaries
- post processing routines in Python

```
// compute the norm of JxB/B**2
__global__ void JxB_B2(REAL *B, REAL *J, REAL *JxB_B2, int dimX, int dimY, int dimZ) {
        int i = threadIdx.x + blockDim.x * blockIdx.x;
        int j = threadIdx.y + blockDim.y * blockIdx.y;
        int k = threadIdx.z + blockDim.z * blockIdx.z;
        int p = threadIdx.x;
        int q = threadIdx.y;
        int r = threadIdx.z;
        int l;
        REAL B2;

        // shared memory for faster communication, the size is assigned dynamically
        extern __shared__ REAL s[];
        REAL *Bs = s;                                   // magnetic field
        REAL *Js = &s[3 * dimX * dimY * dimZ];          // electric current density
        REAL *JxBs = &Js[3 * dimX * dimY * dimZ];       // JxB

        // copy from global memory into shared memory
        if ((i < dev_p.nx) && (j < dev_p.ny) && (k < dev_p.nz)) {
                for (l = 0; l < 3; l++) {
                        Bs[l + p*3 + q*dimX*3 + r*dimX*dimY*3] = B[l + (i+1)*3 + (j+1)*(dev_p.nx+2)*3 + (k+1)*(dev_p.nx+2)*(dev_p.ny+2)*3];
                        Js[l + p*3 + q*dimX*3 + r*dimX*dimY*3] = J[l + i*3 + j*dev_p.nx*3 + k*dev_p.nx*dev_p.ny*3];
                }

                cross(&Js[0 + p*3 + q*dimX*3 + r*dimX*dimY*3],
                      &Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3],
                      &JxBs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3]);

                B2 = dot(&Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3], &Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3]);

                // return result into global memory
                JxB_B2[i + j*dev_p.nx + k*dev_p.nx*dev_p.ny] = norm(&JxBs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3])/B2;
        }
}
```

Similarities with the PencilCode

Fortran name lists

```
&comp
    nx = 33;  ny = 33;  nz = 33
/

&start
    Lx = 0.6;     Ly = 0.6;     Lz = 1.0
    Ox = -0.3;    Oy = -0.3;    Oz = -0.5
    bInit = "sheared"
    ampl = 1.
    initDist = "initShearX"
    initShear0 = 0.7
    initShearK = 1.
    fRestart = t
/
```

Bash commands

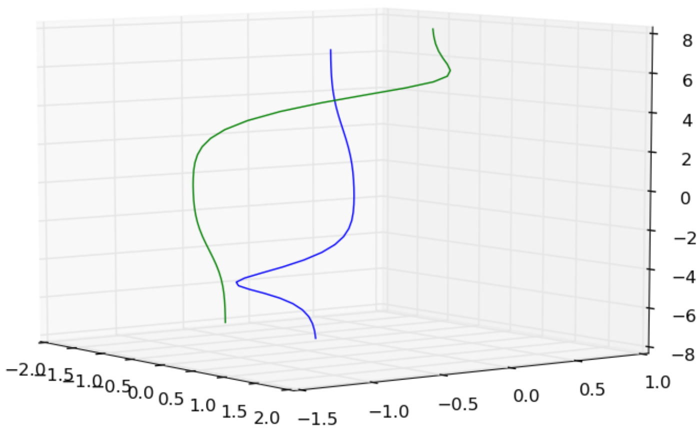
```
gm_ci_run
gm_inspectrun
gm_newrun
```

time_series.dat

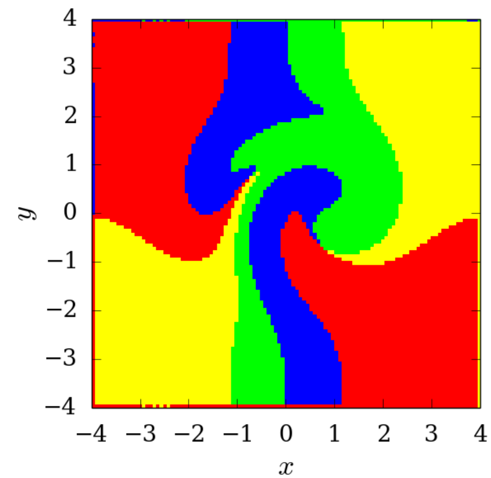
```
#     it           t           dt        maxDelta     JxB_B2Max   epsilonStar          B2       convex
       0  1.29540e-06  1.29540e-06  1.78814e-07  1.50932e+02  2.81160e+02  7.12394e-01 -1.00000e+00
       1  3.08508e-06  1.78967e-06  1.19209e-07  1.13175e+02  2.96738e+02  7.12170e-01 -1.00000e+00
       2  5.76647e-06  2.68139e-06  1.78814e-07  8.83429e+01  3.15884e+02  7.11882e-01 -1.00000e+00
       3  9.47096e-06  3.70449e-06  1.19209e-07  7.67879e+01  3.36120e+02  7.11536e-01 -1.00000e+00
       4  1.50212e-05  5.55028e-06  1.78814e-07  6.44194e+01  3.57402e+02  7.11085e-01 -1.00000e+00
       5  2.13638e-05  6.34253e-06  7.74860e-07  5.44002e+01  3.73753e+02  7.10636e-01 -1.00000e+00
```

Post-Processing

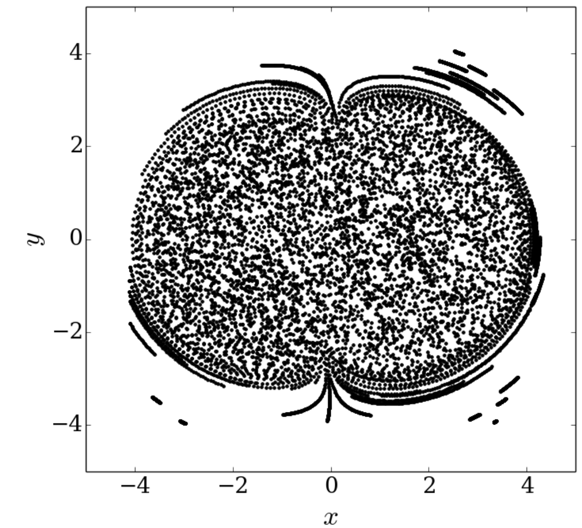
streamlines



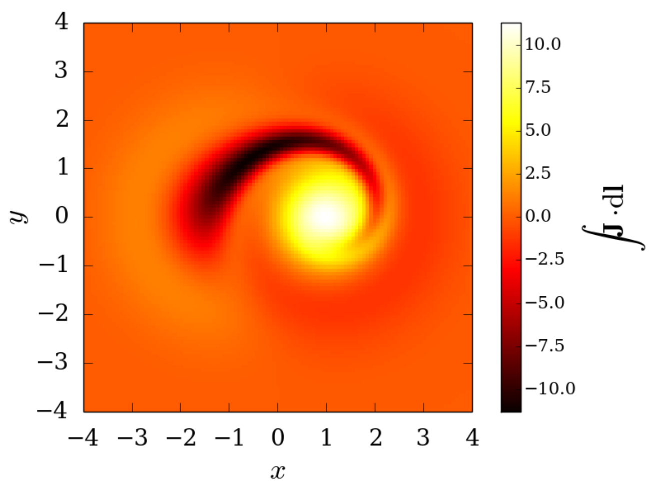
field line mapping



Poincaré maps



line integration



save and read as vtk file

```
s0 = gm.streamInit(tol = 0.01)
```

⇩

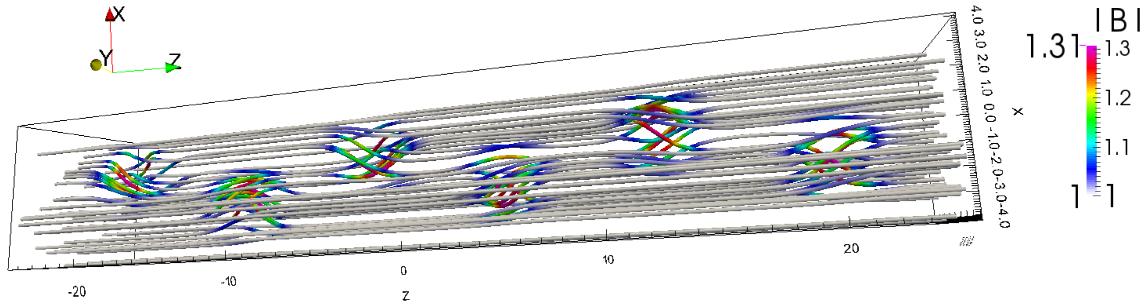
```
stream.vtk
```

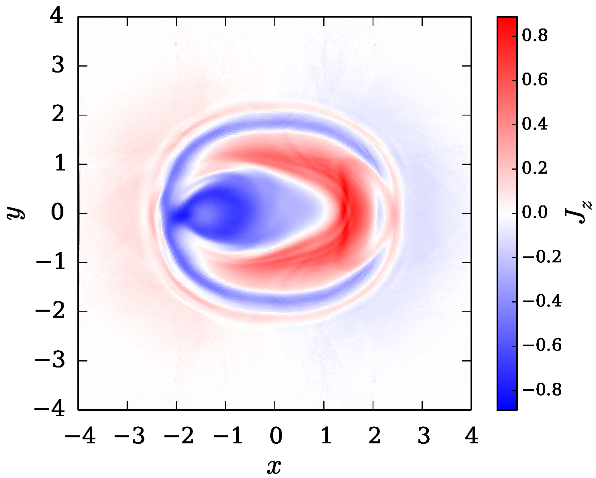
⇩

```
sr = gm.readStream()
```

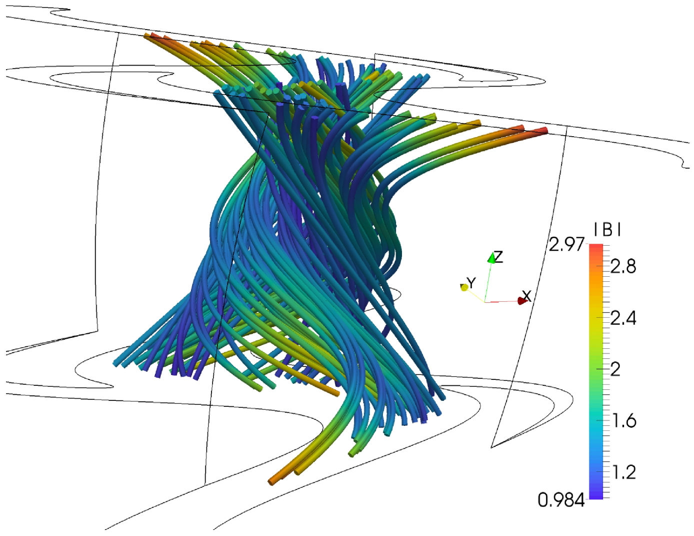
Current Structures

Braided fields:

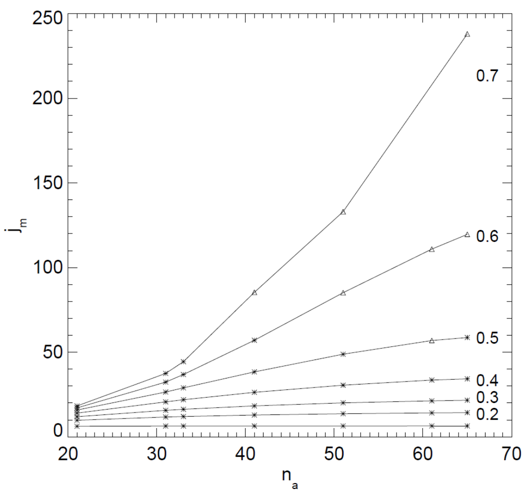




Sheared fields:

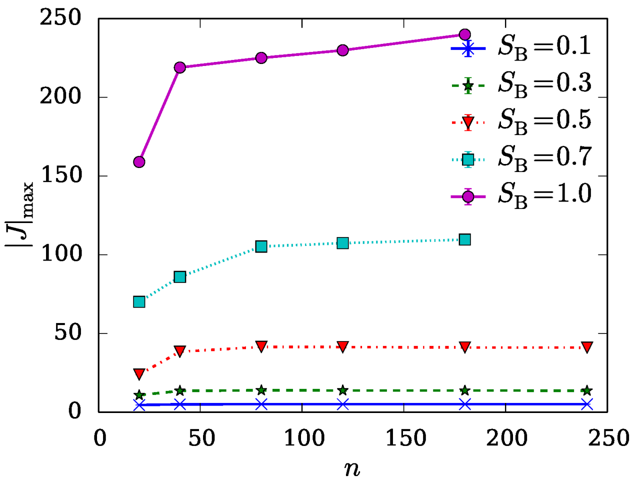


then:



Longbottom (1998)

now:



Conclusions

- Lagrangian numerical scheme for ideal evolution.
- Preserving field line topology.
- Mimetic methods more capable of producing force-free fields.
- GLEMuR code running on GPUs.
- No evidence for current sheets in sheared and braided setups

simon.candelaresi@gmail.com

www.maths.dundee.ac.uk/scandelaresi/

www.youtube.com/user/iomsn/videos