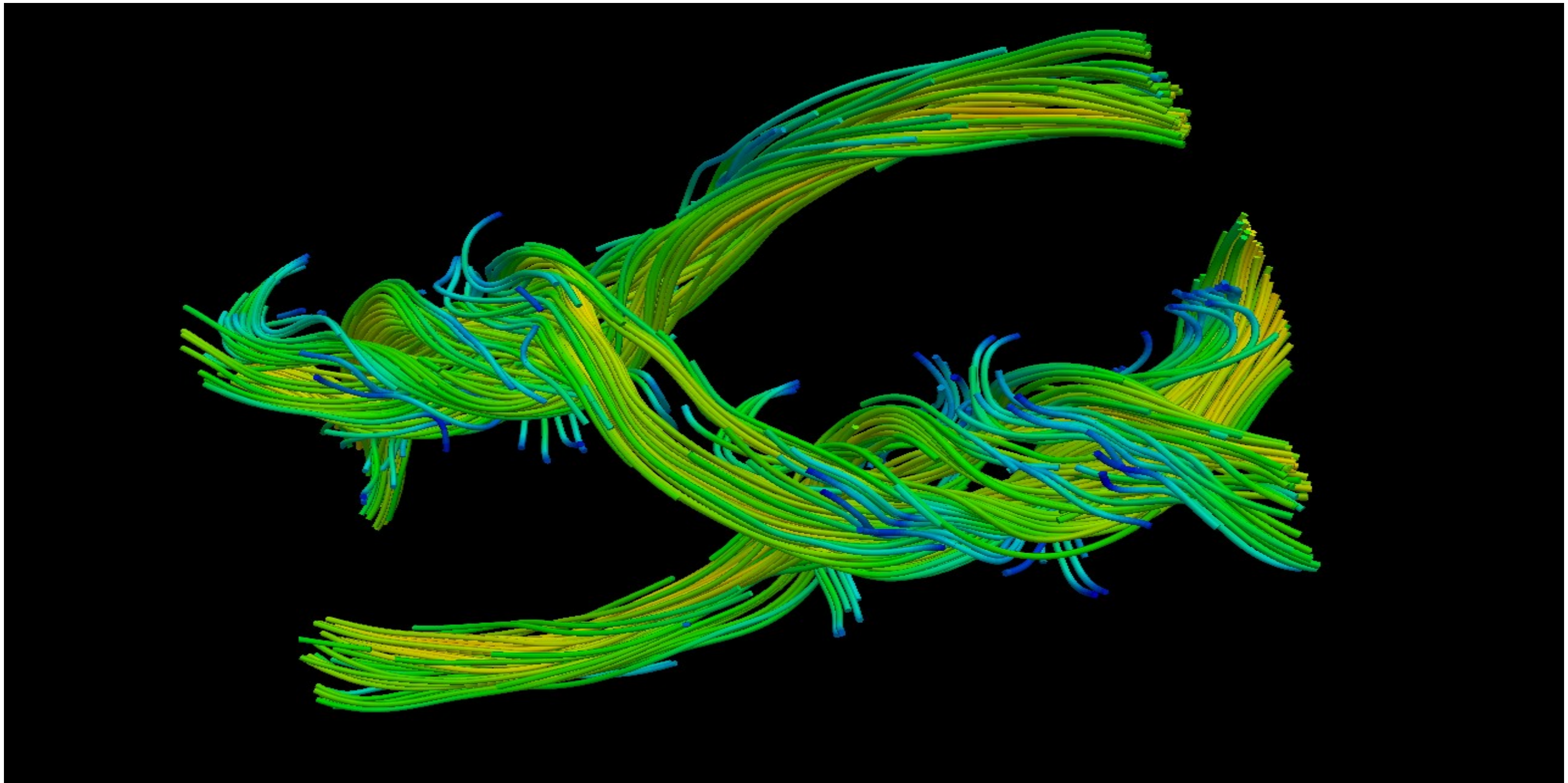
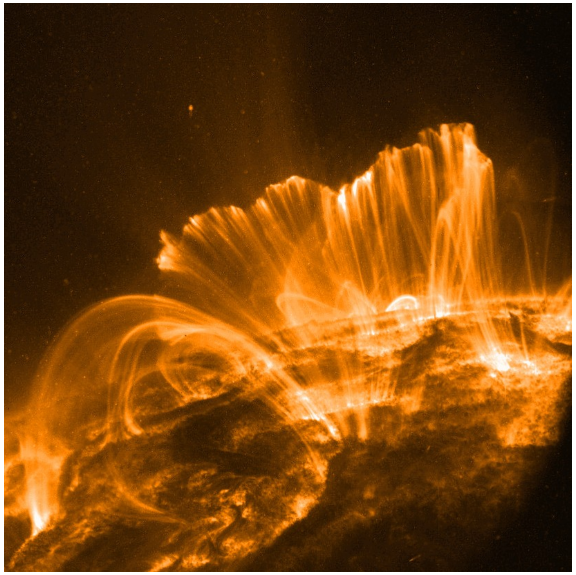
Methods for Quantifying Magnetic Field Topology

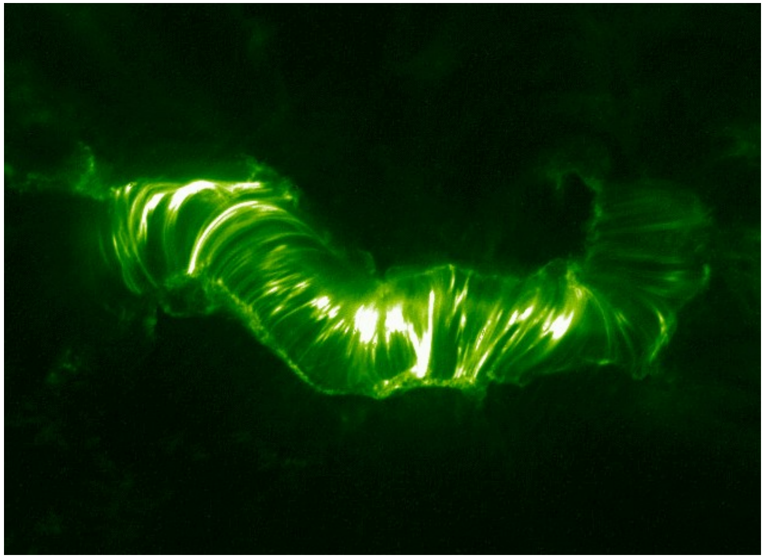
Simon Candelaresi



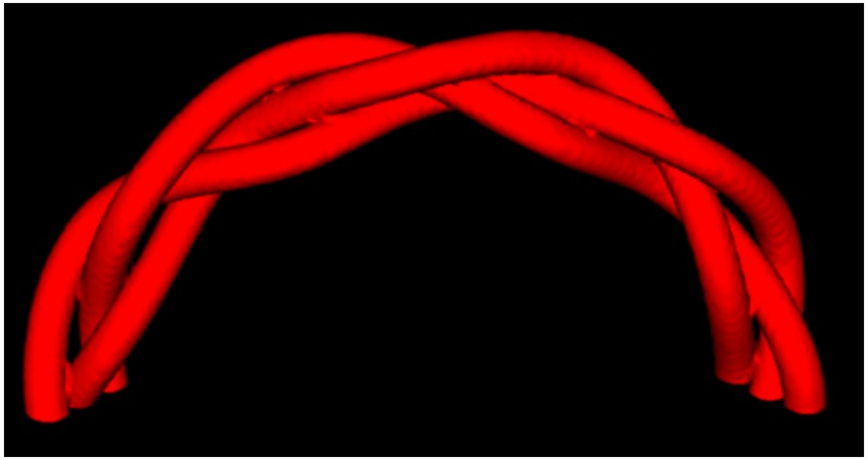
Solar Magnetic Field



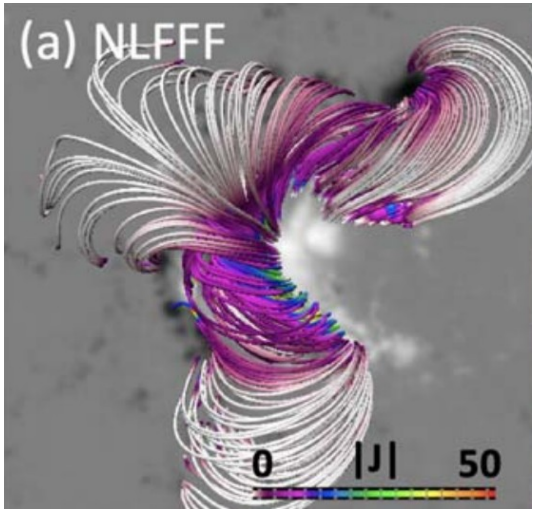
(Trace)



(Trace)



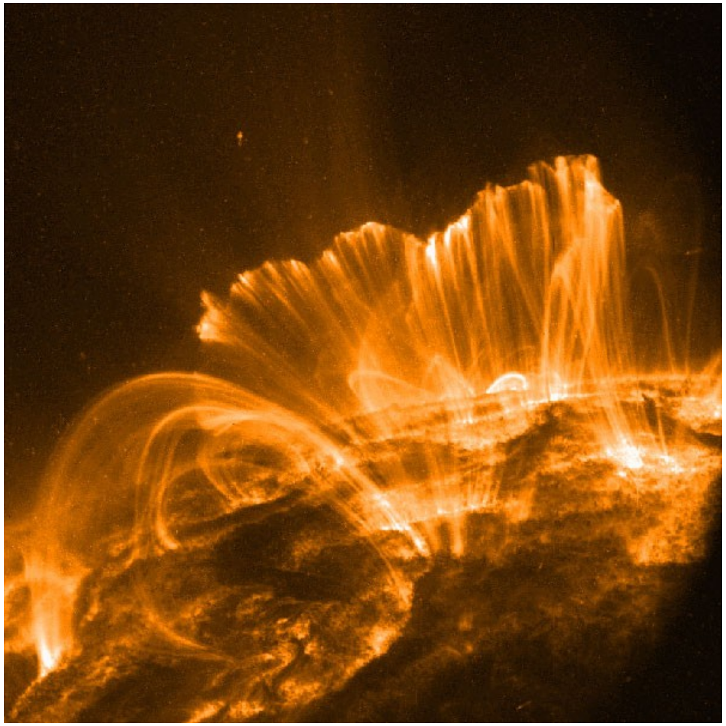
(Prior and MacTaggart 2016)

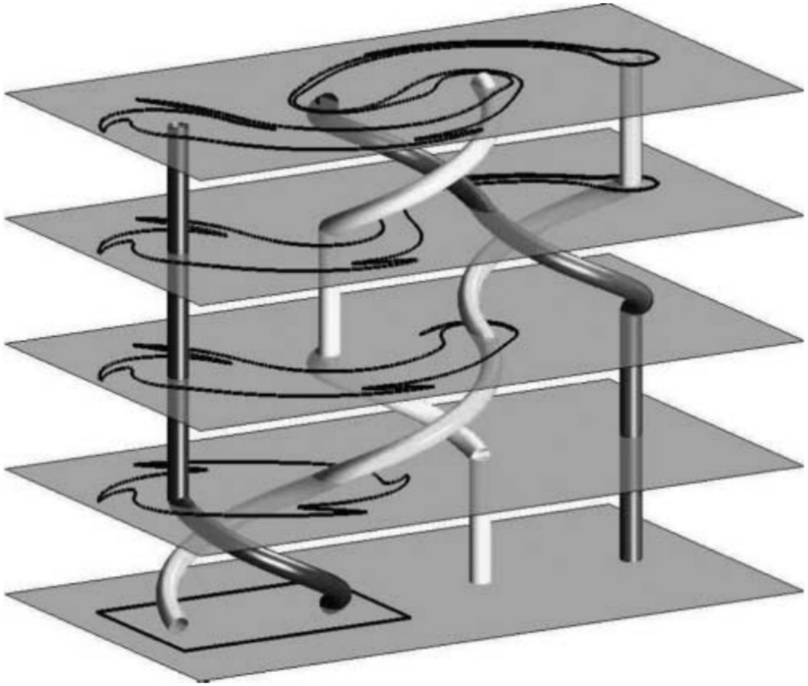


(Yamasaki et al. 2021)

Coronal Magnetic Fields

NASA





(Thiffeault et al. 2006)

Field line tangling in solar magnetic fields.

Magnetic Helicity

$$H_{\mathrm{m}} = \int \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Conservation of magnetic helicity:

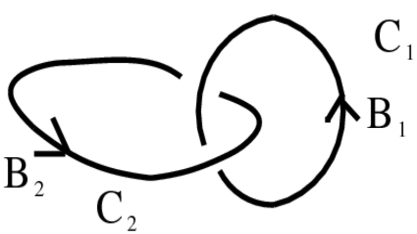
$$\frac{\partial}{\partial t} \int_V \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V = -2\eta \int_V \mathbf{J} \cdot \mathbf{B}$$

$\eta =$ magnetic resistivity

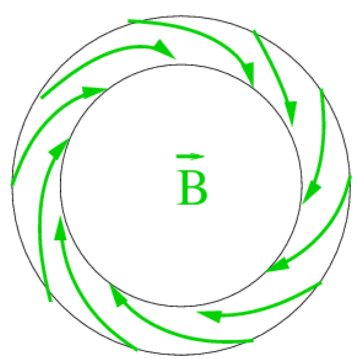
Realizability condition:

$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

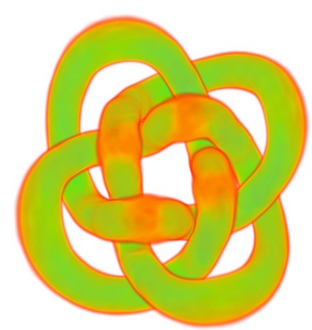
Magnetic energy is bound from below by magnetic helicity.



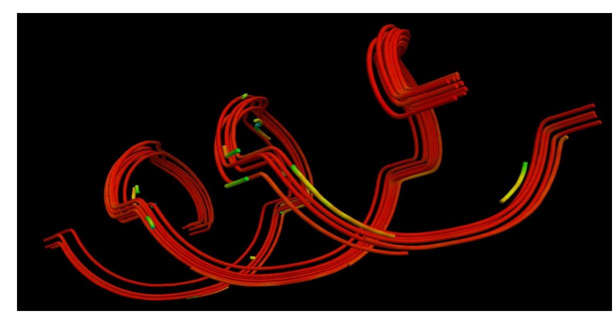
link



twist



knot

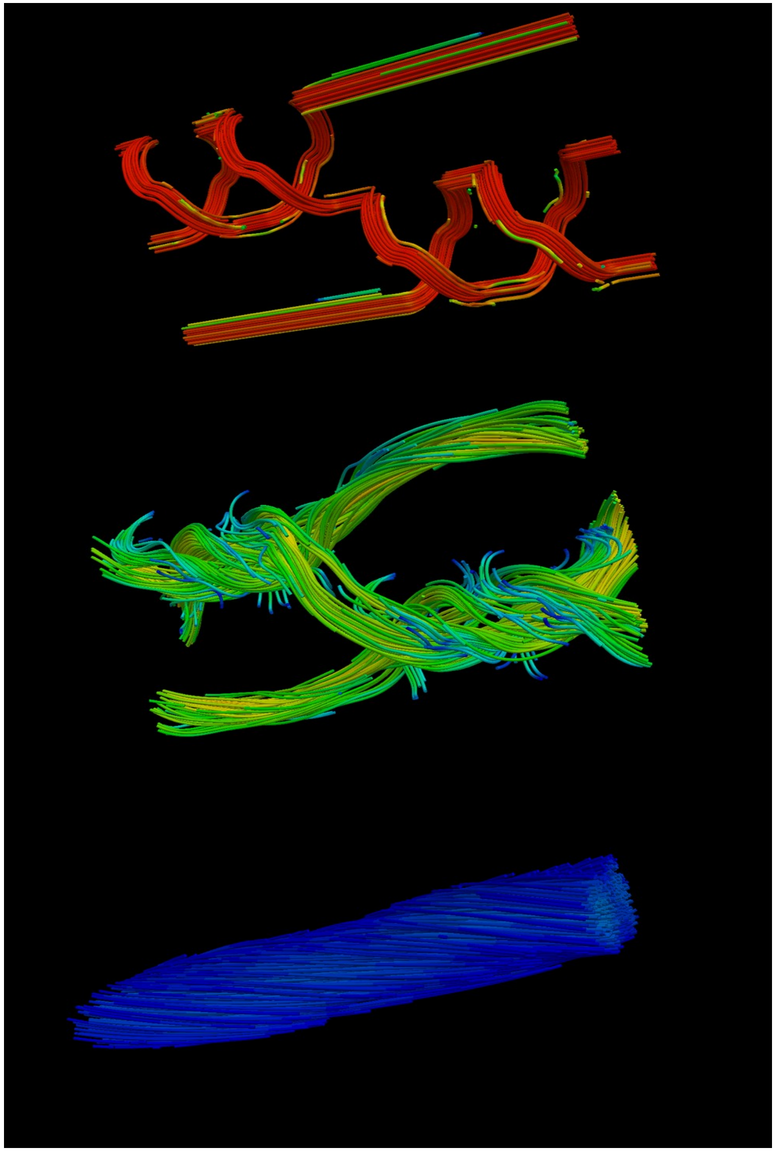


braid

Magnetic Braid Configurations

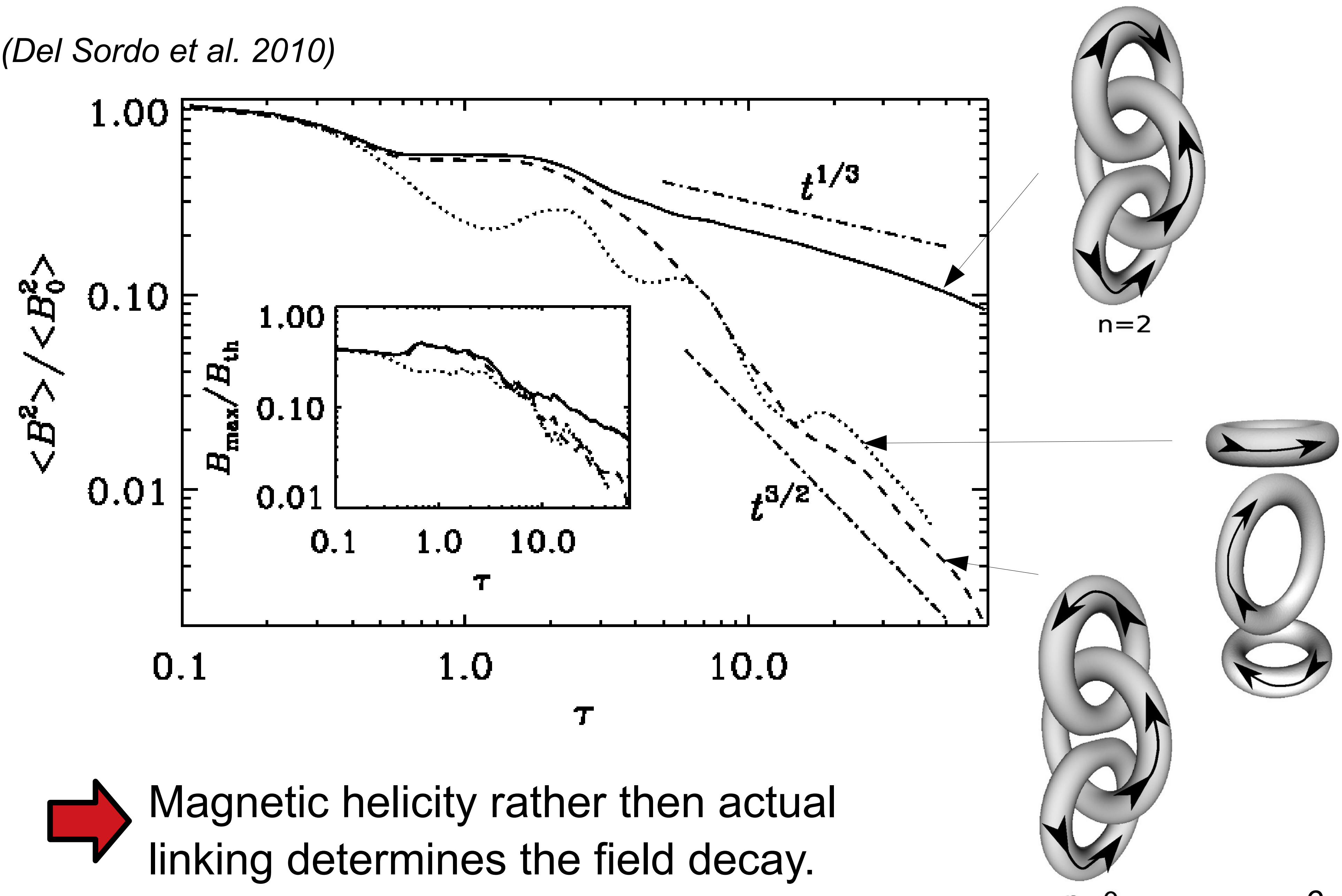
AAA (trefoil knot)

AABB (Borromean rings)



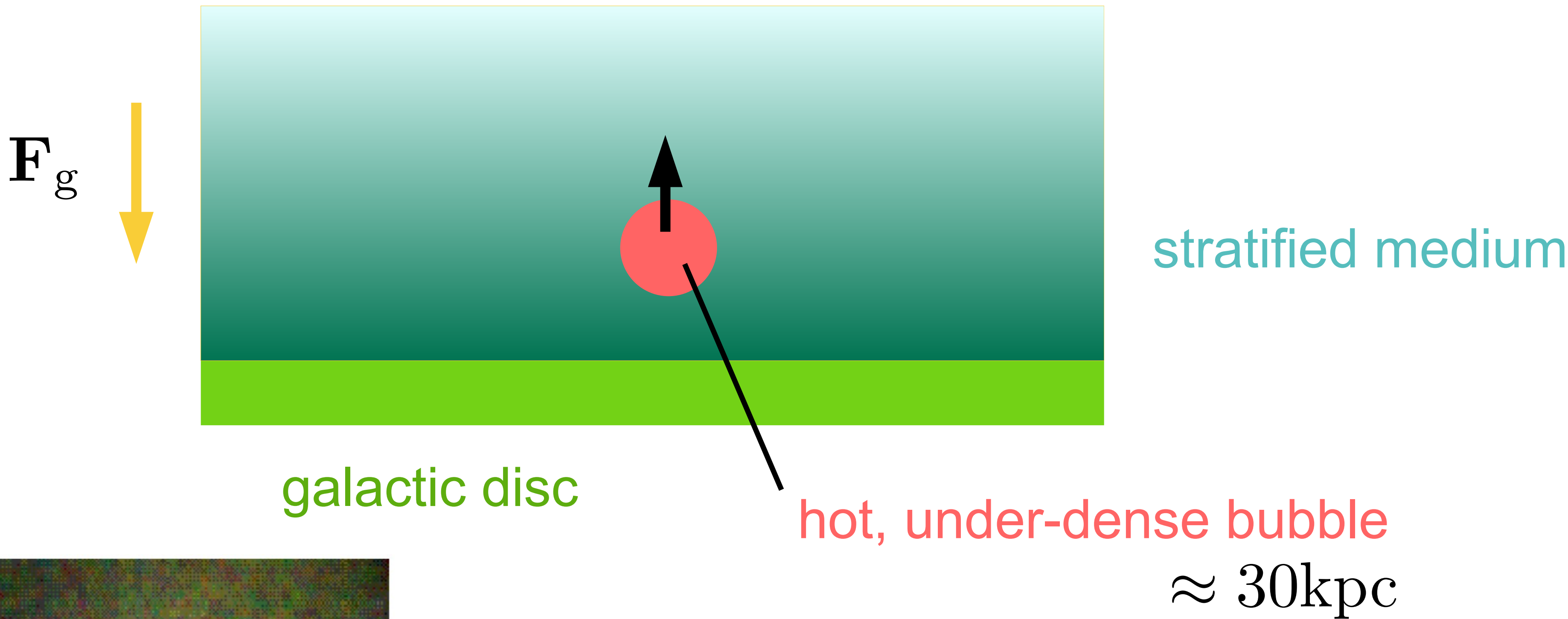
Interlocked Flux Rings

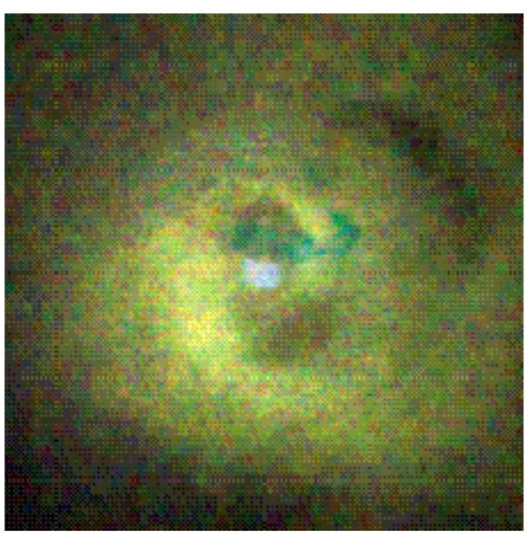
(Del Sordo et al. 2010)



Magnetic helicity rather then actual linking determines the field decay.

Intergalactic Bubbles



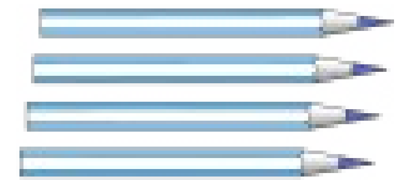


(Fabian et al. 2000)

- Bubbles rise buoyantly through density difference.
- Bubbles' age is several tens of millions of years.

Numerical Experiments

Full resistive magnetohydrodynamics simulations with the Pencil Code.

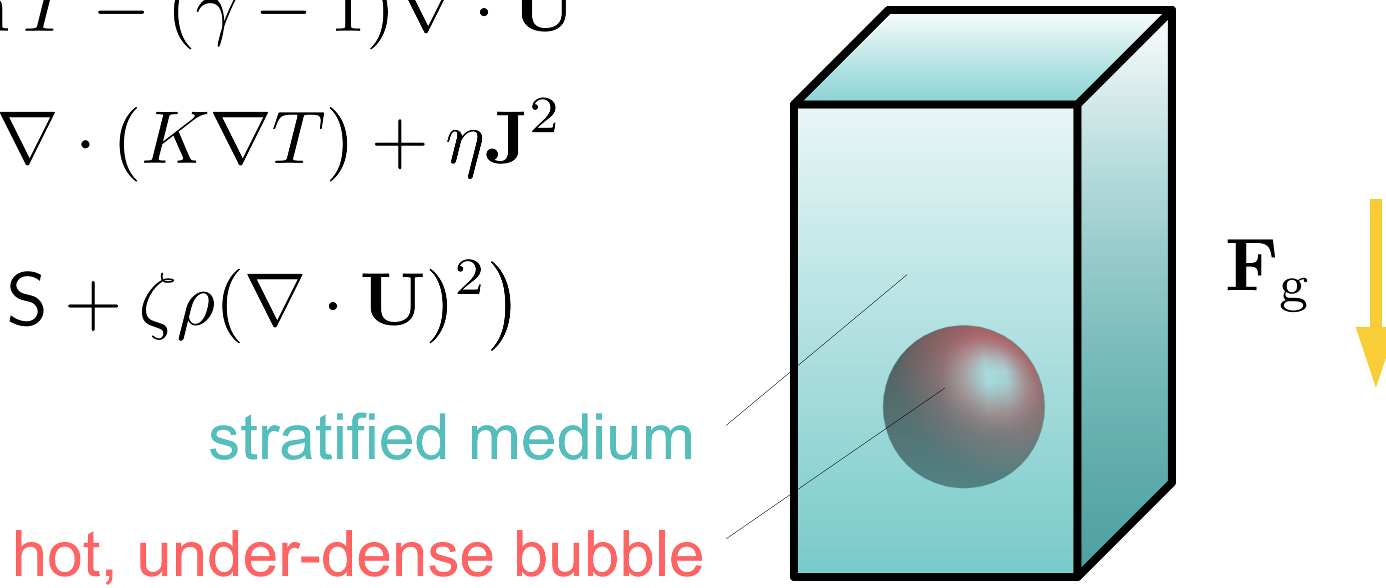


$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \eta \nabla^2 \mathbf{A}$$

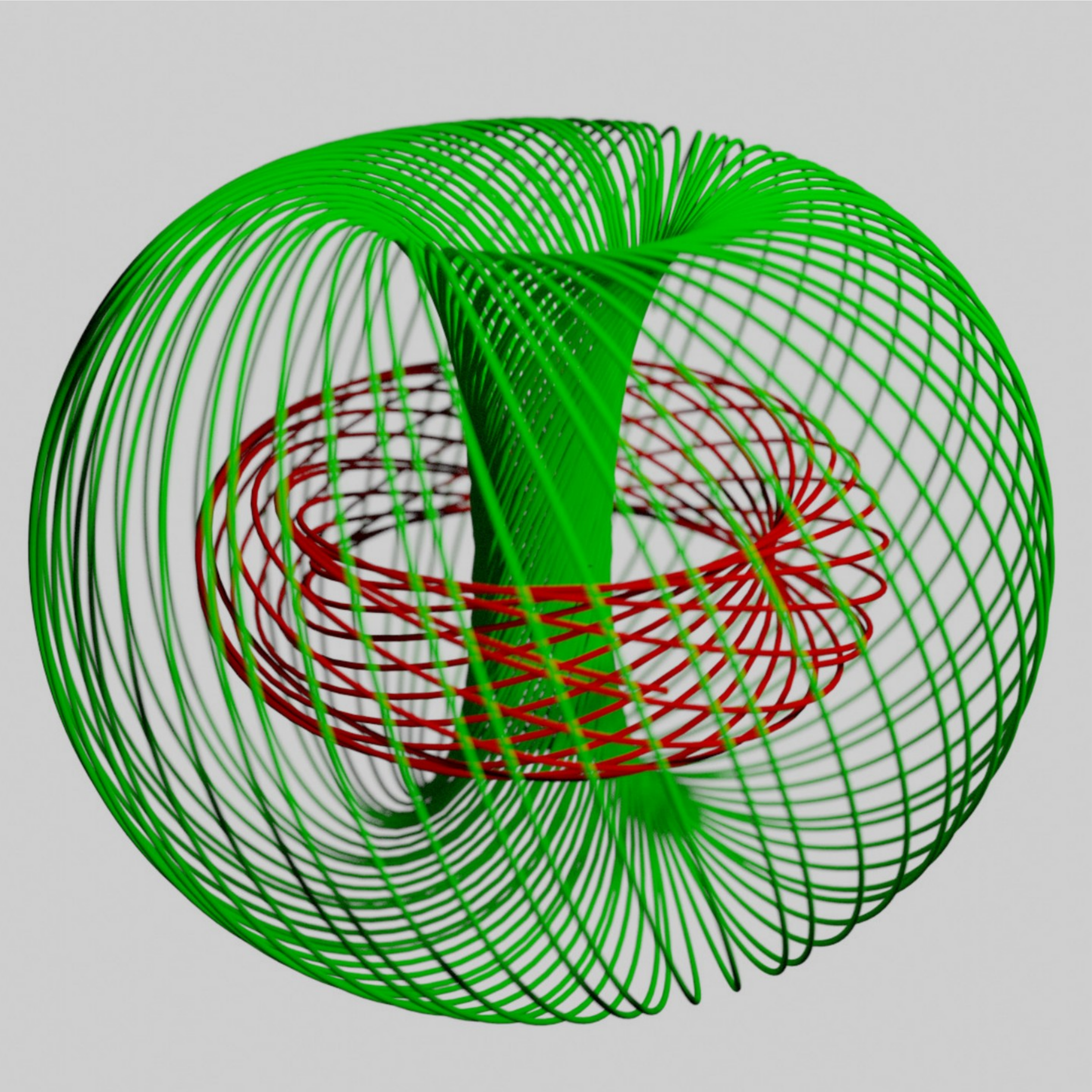
$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \nabla \left(\frac{\ln T}{\gamma} \ln \rho \right) + \mathbf{J} \times \mathbf{B}/\rho - \mathbf{g} + \mathbf{F}_{\mathrm{visc}}$$

$$\frac{\partial \ln T}{\partial t} = -\mathbf{U} \cdot \nabla \ln T - (\gamma - 1) \nabla \cdot \mathbf{U} + \frac{1}{\rho c_V T} \left(\nabla \cdot (K \nabla T) + \eta \mathbf{J}^2 + 2\rho\nu \mathsf{S} \otimes \mathsf{S} + \zeta \rho (\nabla \cdot \mathbf{U})^2 \right)$$

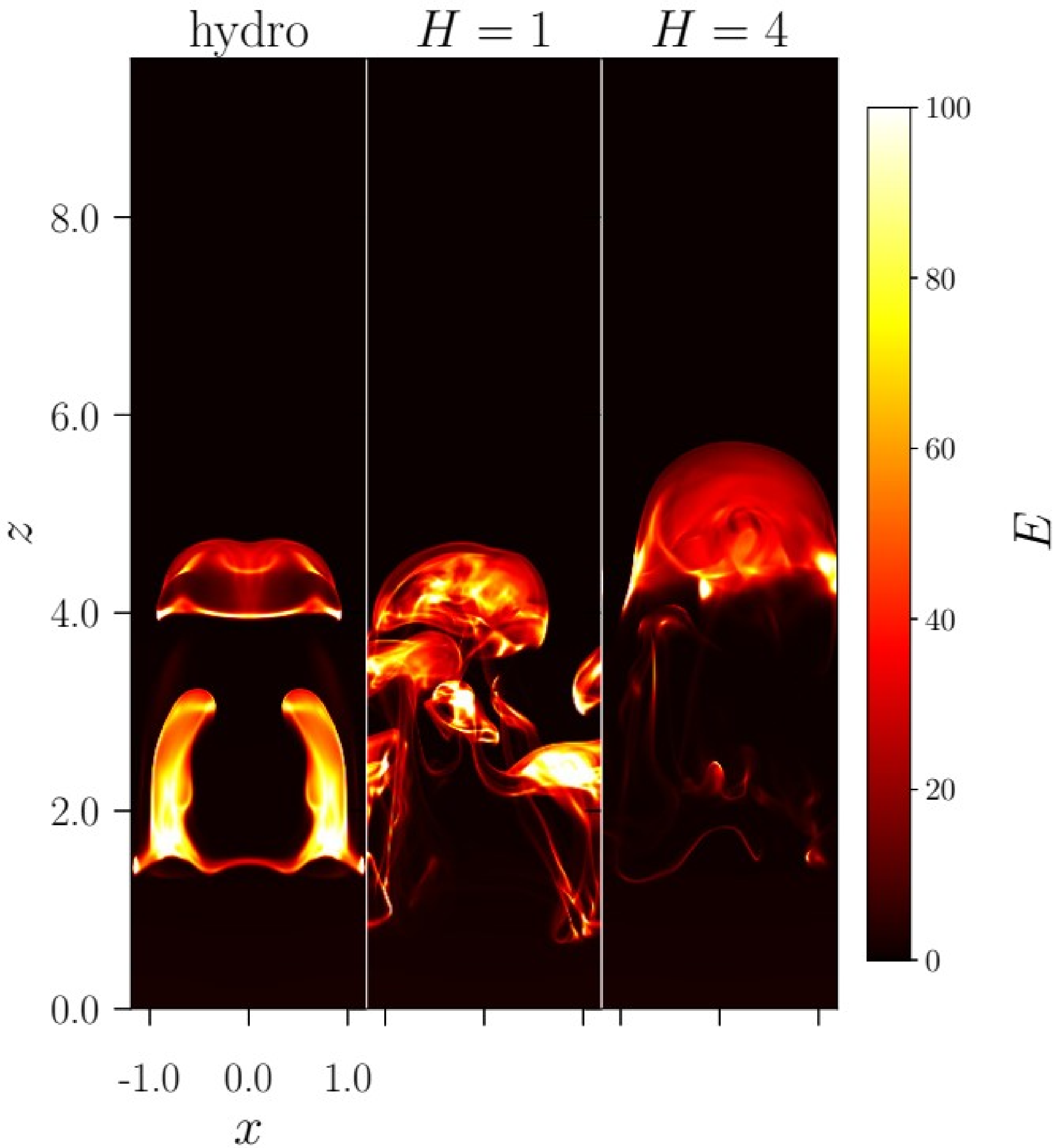
$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\nabla \cdot \mathbf{U}$$



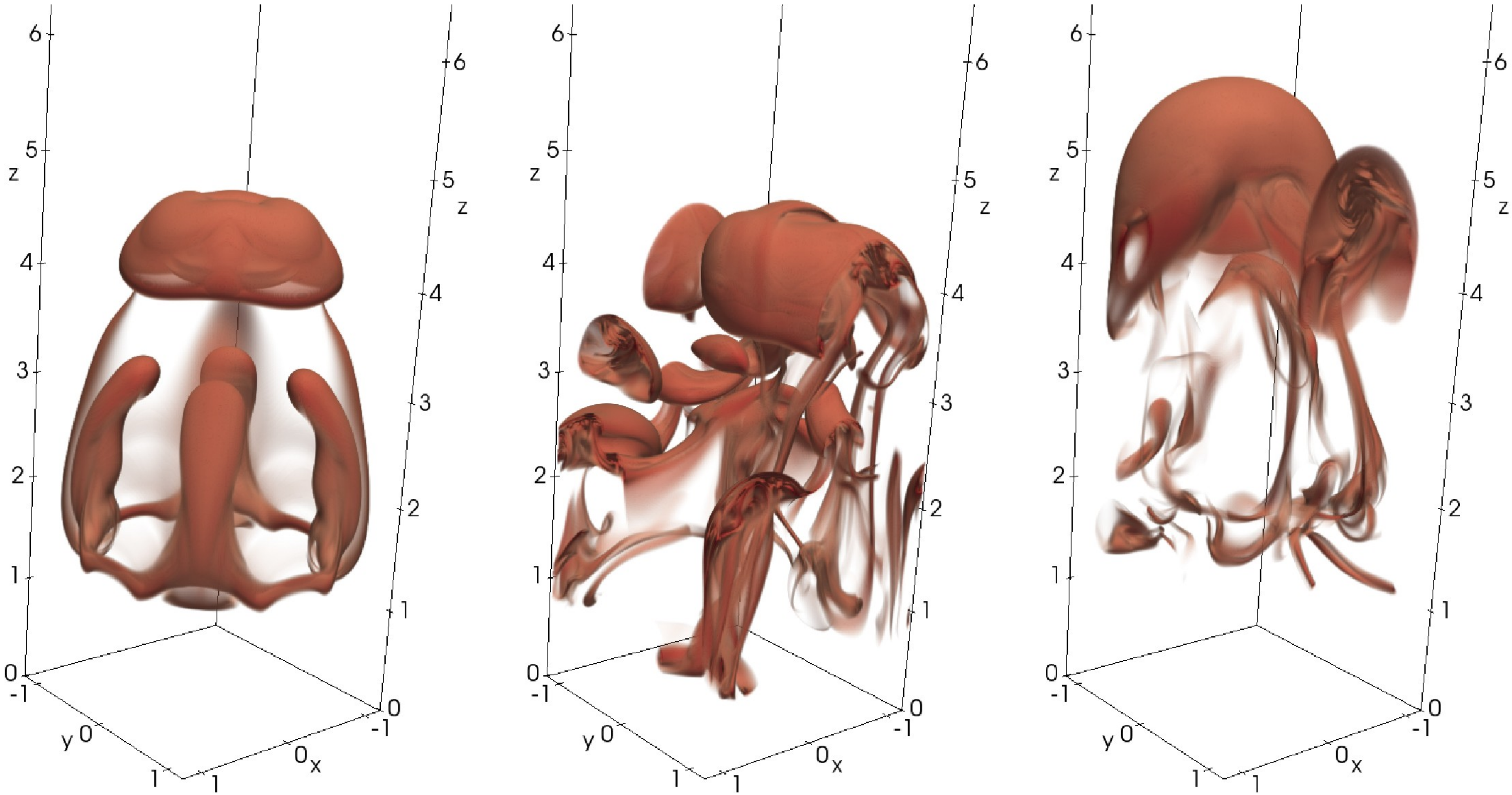
Initial Condition: Spheromak



Thermal Emission

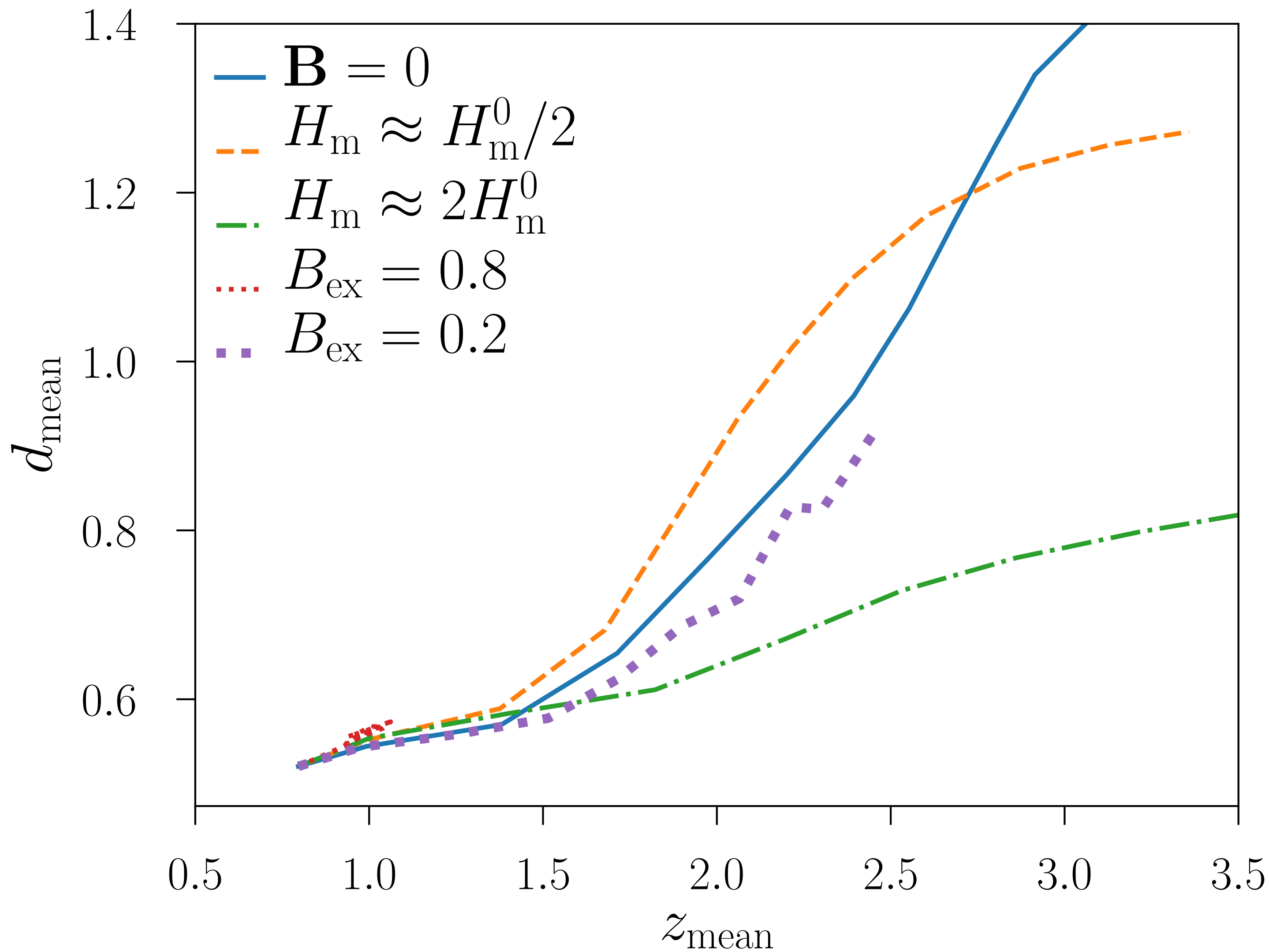


Temperature Iso-Surfaces



hydro | low helicity | high helicity

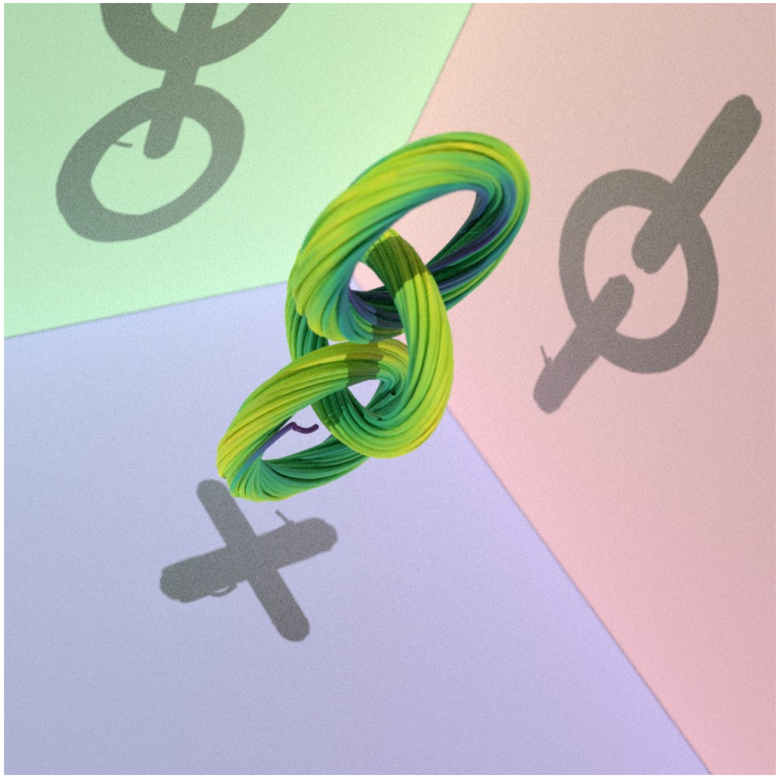
Bubble Coherence



Helical magnetic fields can stabilise the bubbles.

Magnetic Fields with a Twist

- Helical fields can be made non-helical by twisting the field lines.
- Non-helical fields can be made helical by twisting the field lines.
- Simulated twisted knots and links in MHD (Pencil Code).

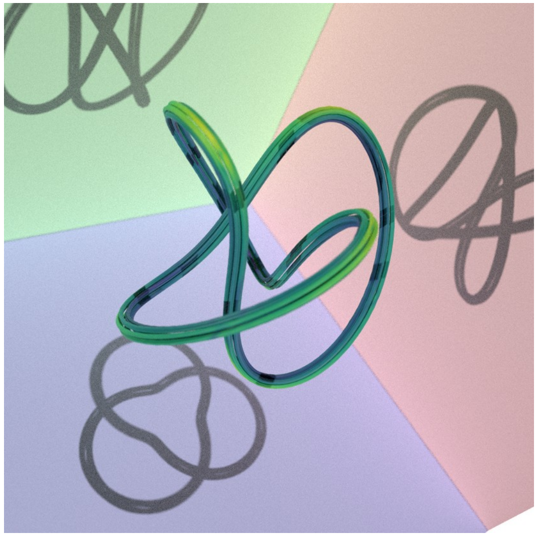


(Candelaresi & Beck 2023)

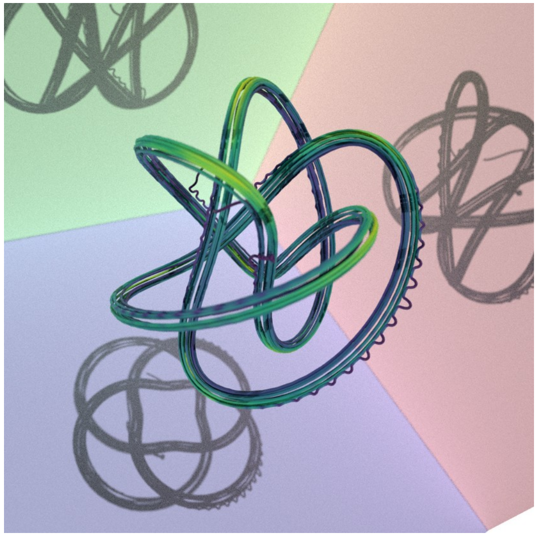
$$E_{\mathrm{M}}(t) = ? \qquad \frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{m}} = ? \qquad \int_V \mathbf{J} \cdot \mathbf{B} \, \mathrm{d}V = ?$$

Knots and Links

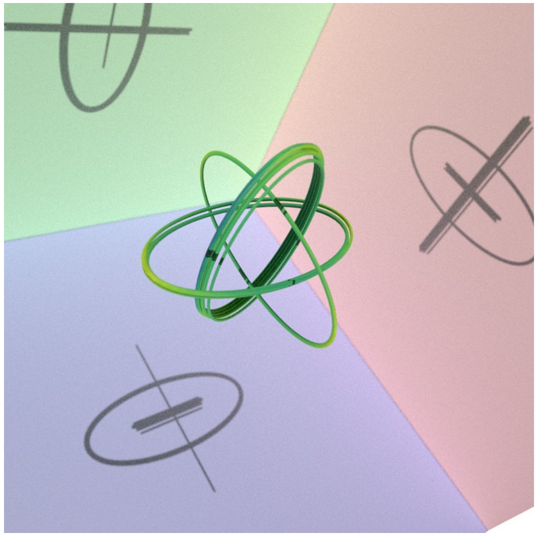
trefoil



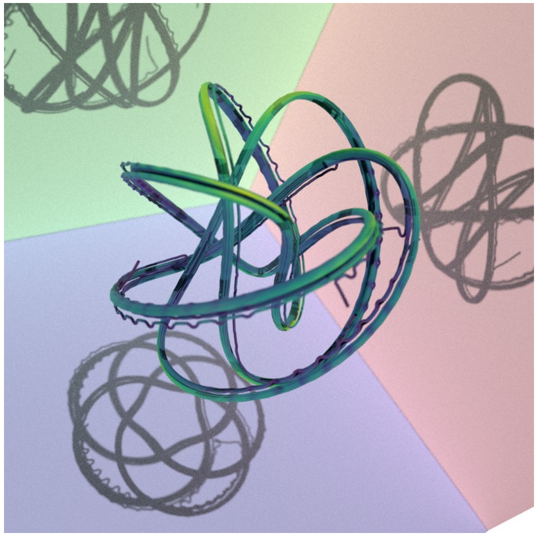
4-foil



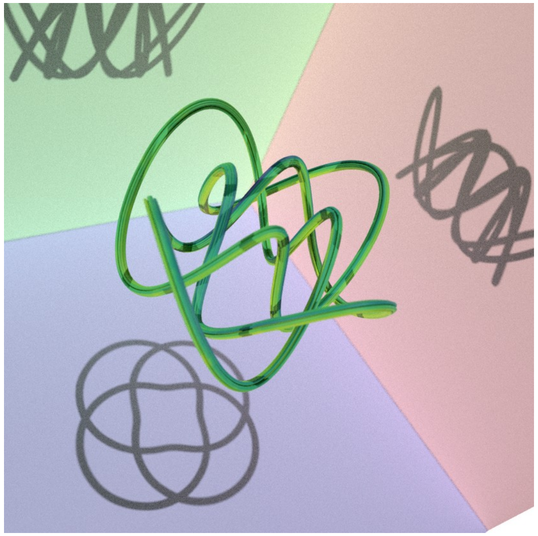
Borromean rings



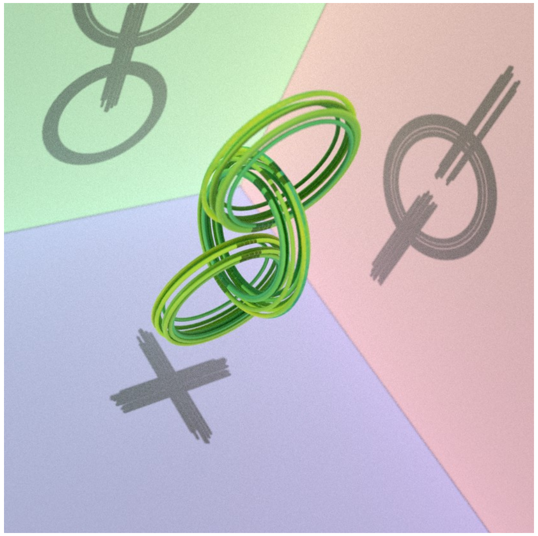
5-foil



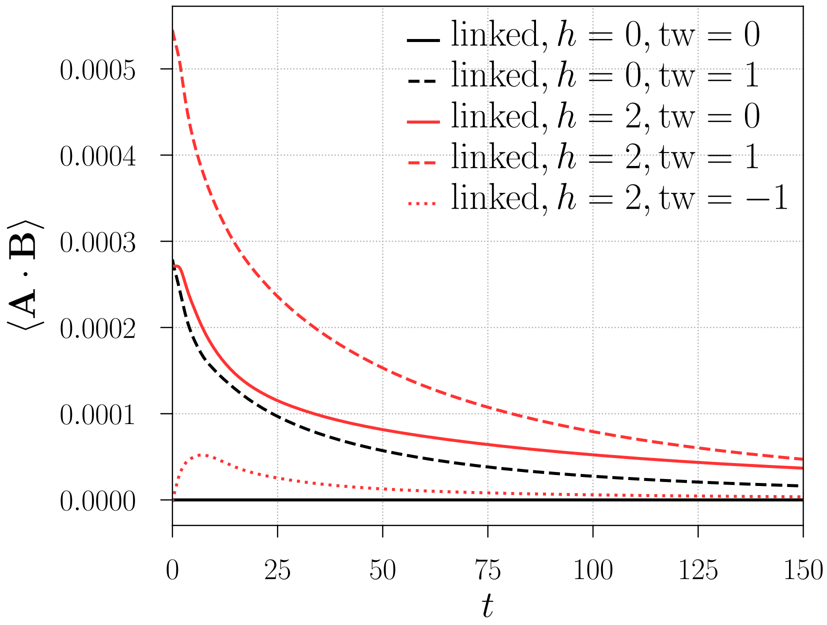
IUCAA (8_18)

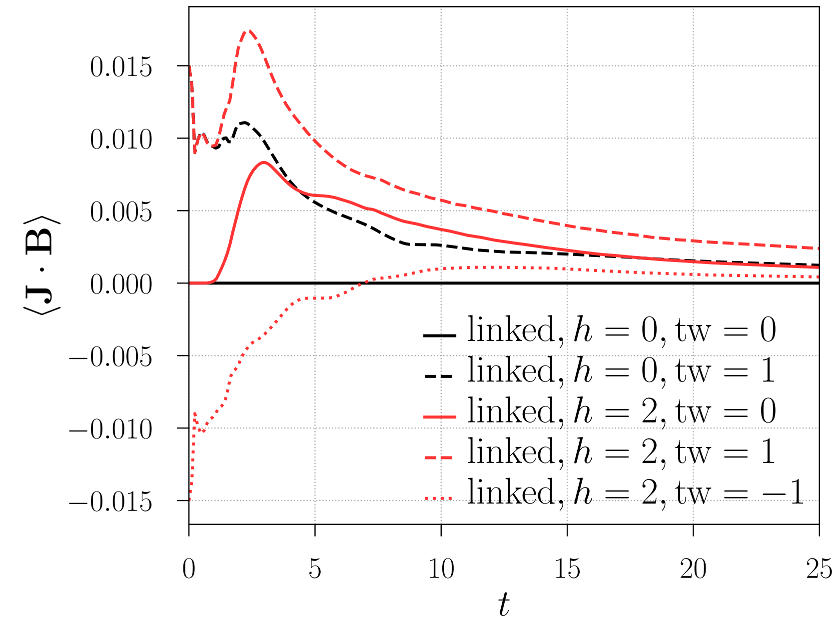


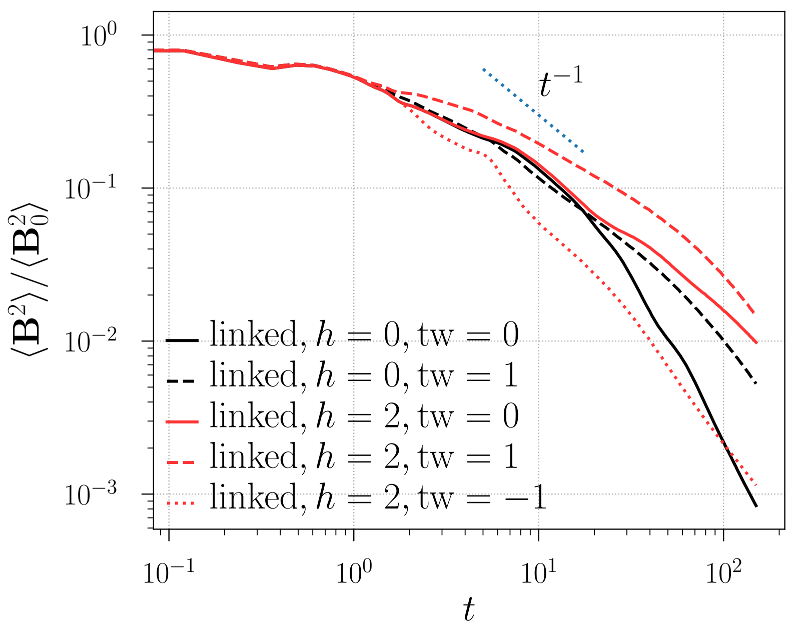
triple rings



Triple Rings

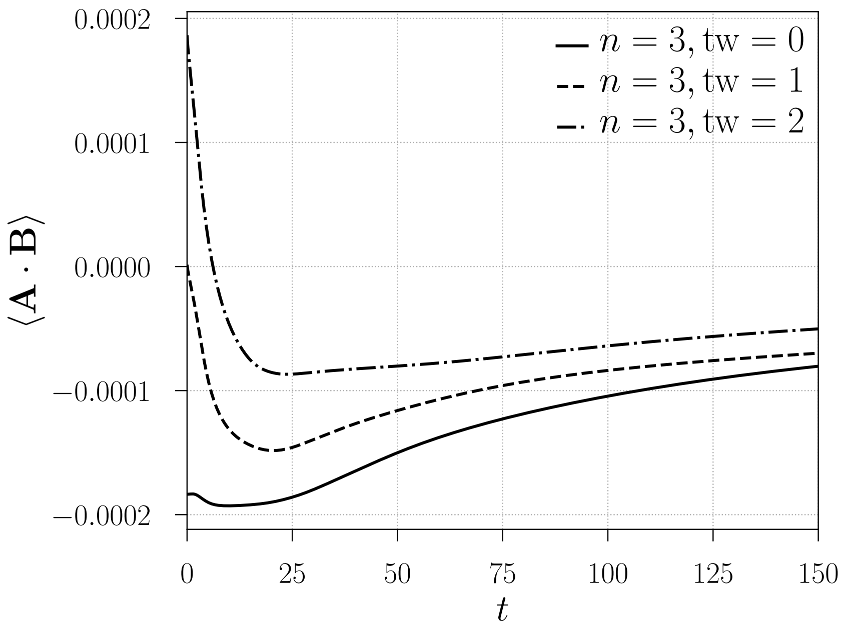


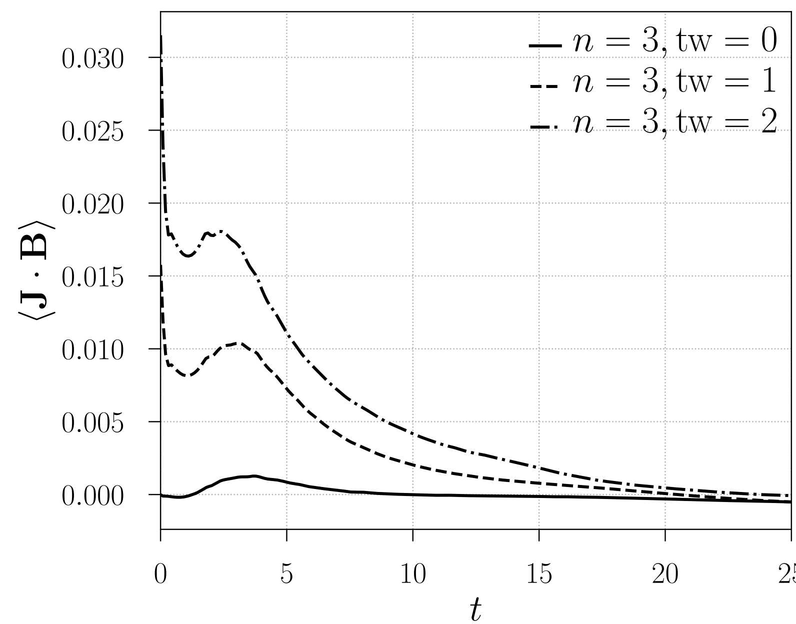


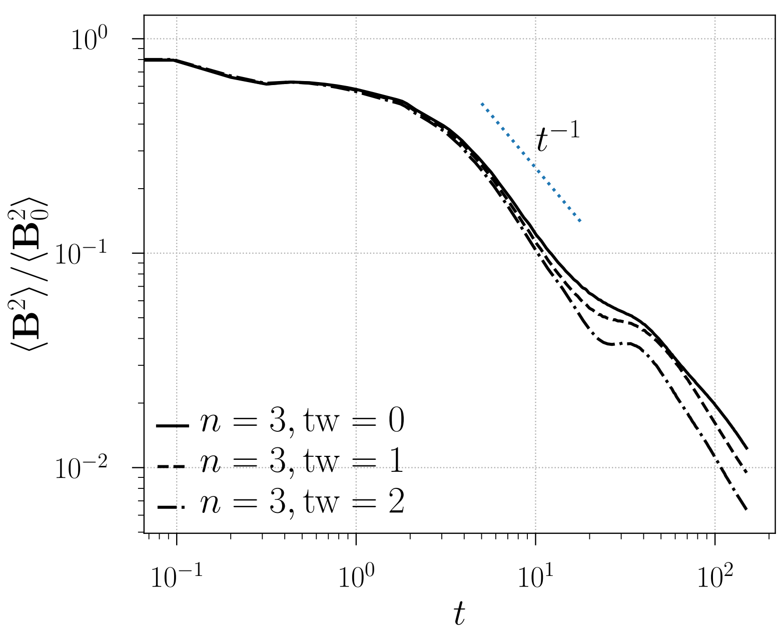


- Helicity restricts decay.
- Small helicity production in twisted non-helical field.

Knots



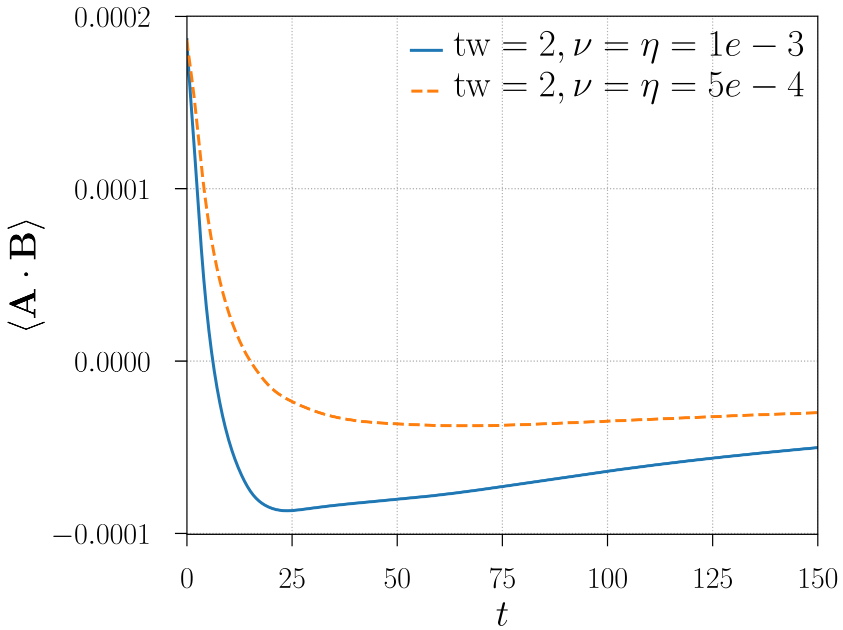


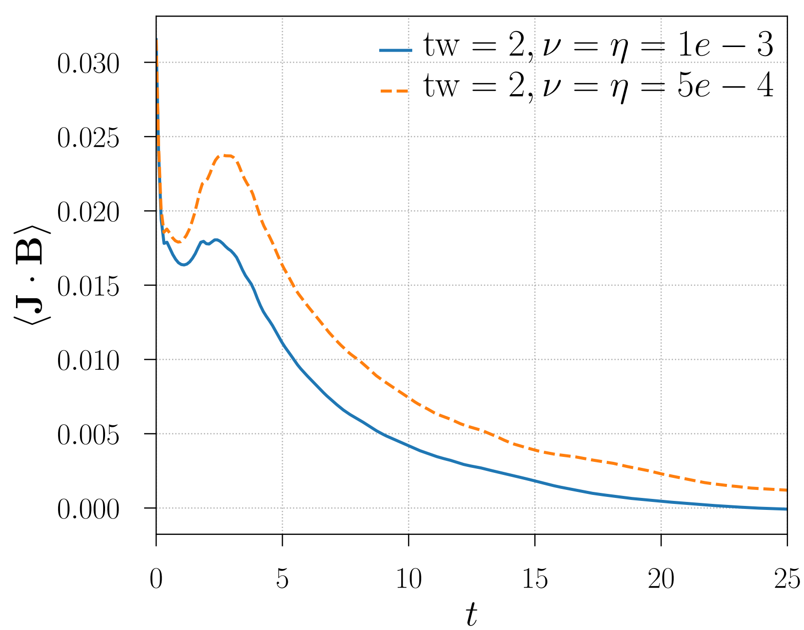


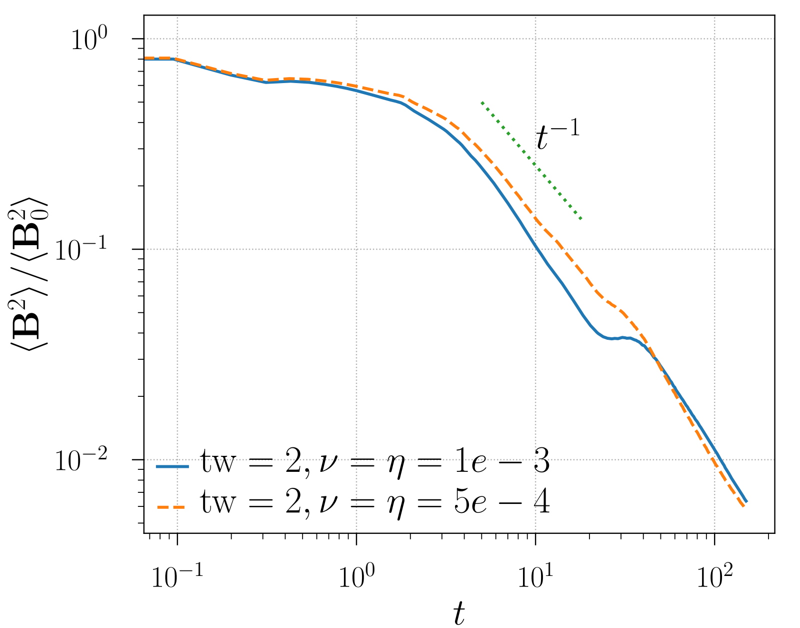
Significant helicity production.

Initial helicity is not a good predictor on dynamics.

Low Resistivity Twisted Trefoil Knot

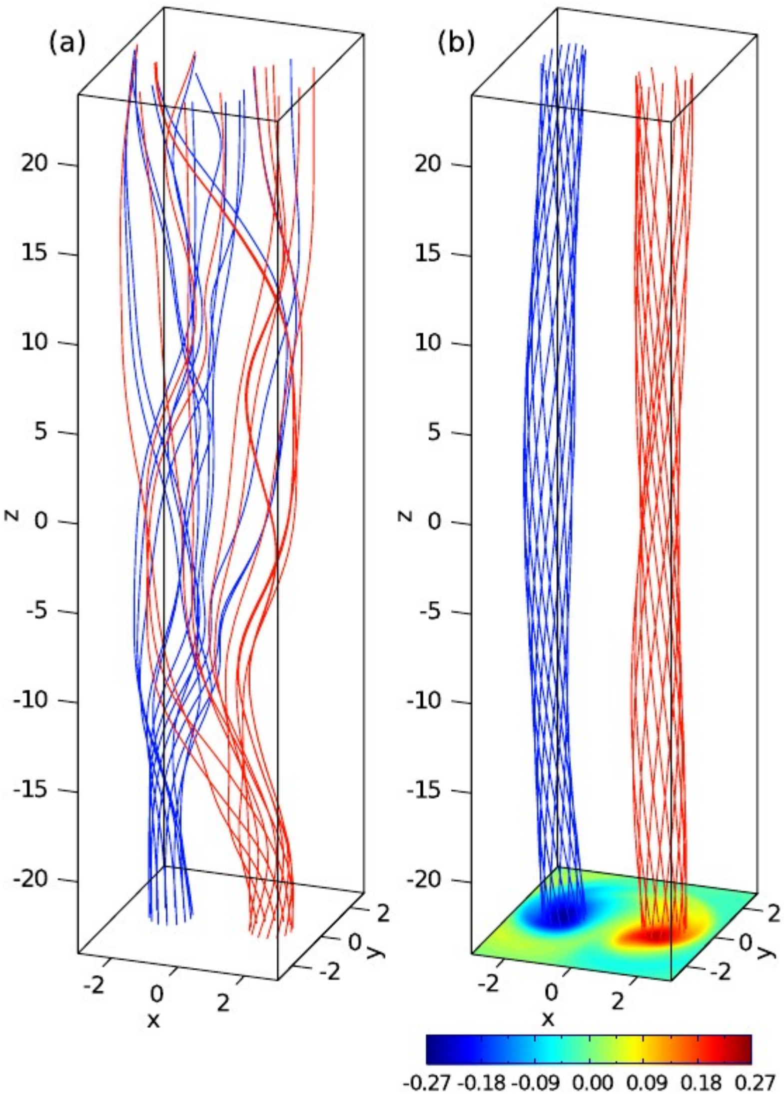




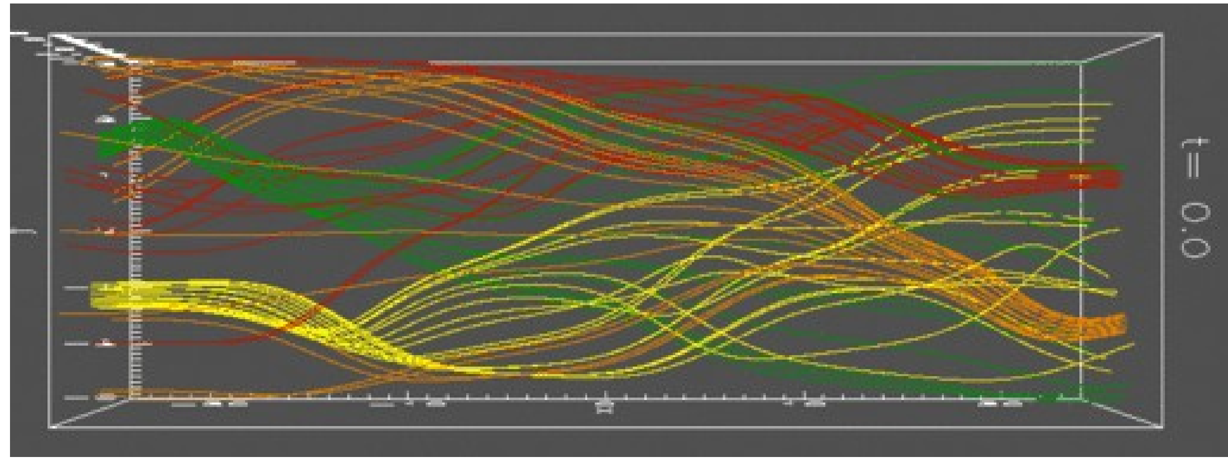


- Stronger alignment of J and B.
- Lower resistivity partially compensated by stronger alignment.

Magnetic Braid



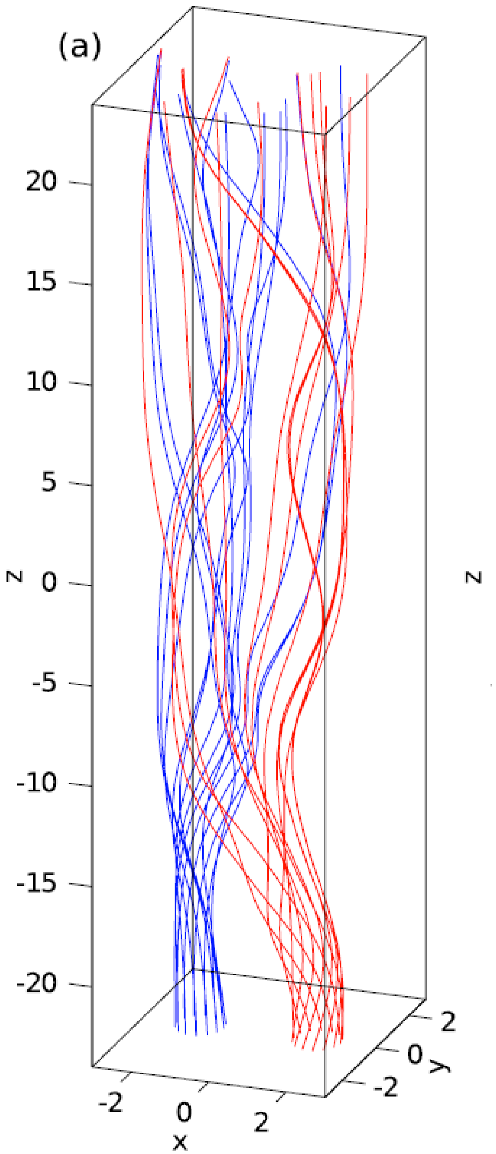
(Yeates et al. 2011)



(Wilmot-Smith 2010)

- Periodic braid topologically equivalent to Borromean rings.
- Separation into two twisted field regions.
- Conserved invariants like fixed point index and field line helicity.

Fixed Point Index



(Yeates et al. 2011)

Trace magnetic field lines from z_0 to z.

mapping: $(x, y) \rightarrow \mathbf{F}_z(x, y)$

fixed points: $\mathbf{F}_1(x, y) = (x, y)$

Color coding:

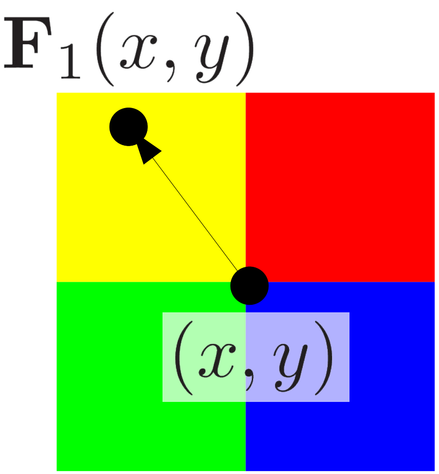
Compare (x, y) with $\mathbf{F}_1(x, y)$:

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y > y$ → red

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y > y$ → yellow

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y < y$ → green

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y < y$ → blue



Stability criteria

constraint | equilibrium

Woltjer (1958): $$\frac{\partial}{\partial t}\int_V \mathbf{A}\cdot\mathbf{B}\;\mathrm{d}V = 0 \qquad \boldsymbol{\nabla}\times\mathbf{B} = \alpha\mathbf{B}$$

Taylor (1974): $$\frac{\partial}{\partial t}\int_{\tilde{V}} \mathbf{A}\cdot\mathbf{B}\;\mathrm{d}V = 0 \qquad \boldsymbol{\nabla}\times\mathbf{B} = \alpha(a,b)\mathbf{B}$$

constant along field line

V = total volume $\tilde{V}$ = volume along magnetic field line

➡ Taylor state not reached due to fixed point conservation.

(Yeates et al. 2011)

Quadratic Helicities

$$\chi^{(2)}(\mathbf{B}) = \sum_{i,j,k} \frac{\Phi_i^2 \Phi_j \Phi_k n(L_j, L_i) n(L_i, L_k)}{(\Omega_i)}$$

$$\chi^{[2]}(\mathbf{B}) = \sum_{i,j} \frac{\Phi_i^2 \Phi_j^2 n^2(L_i, L_j)}{(\Omega_i)(\Omega_j)}$$

$n(L_i, L_j) =$ number of mutual linking

$\Phi_i =$ magnetic flux

$\Omega_i =$ volume of the flux tube

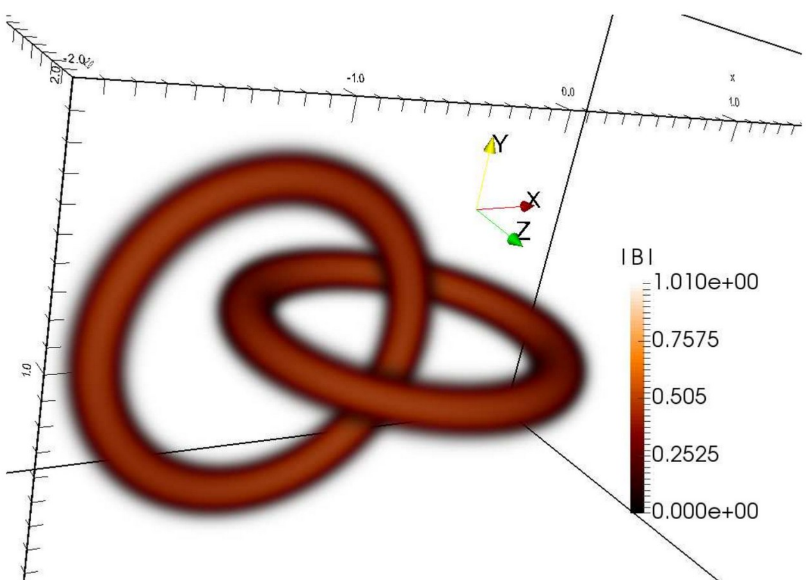
These are not invariant under general diffeomorphisms.
Only under volume preserving ones.

Quadratic Helicities

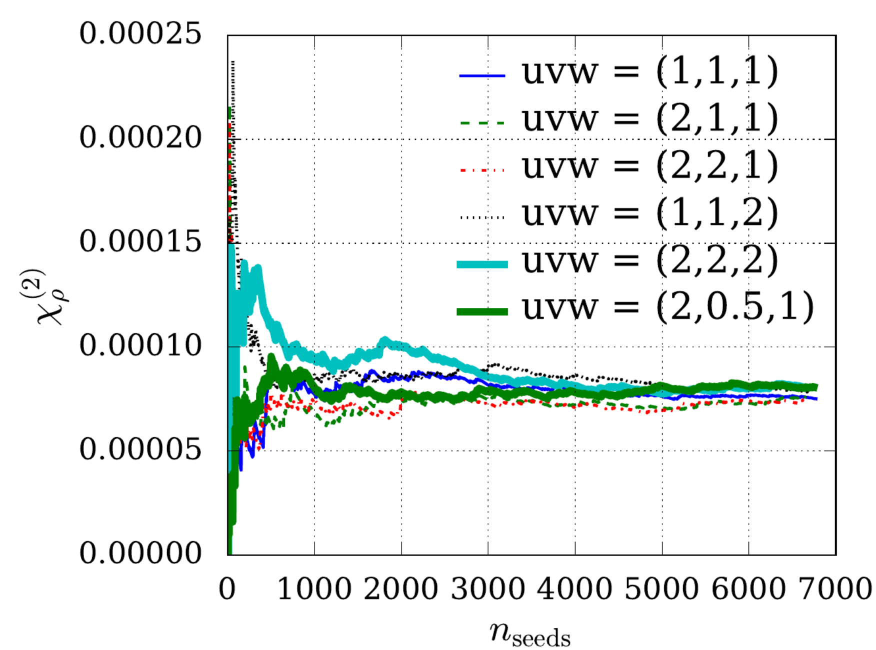
$$\chi_\rho^{(2)} = \iint \Lambda_\rho^2 \rho \ D$$

$$\Lambda_\rho^{(2)}(T;\mathbf{x}) = \frac{1}{T^2}\left(\int_0^T \frac{(\dot{\mathbf{x}}(\tau), \mathbf{A}(\mathbf{x}(\tau)))}{\rho(\mathbf{x}(\tau)))} \ \tau\right)^2$$

★ Invariant under homogeneously density changing diffeomorphisms.

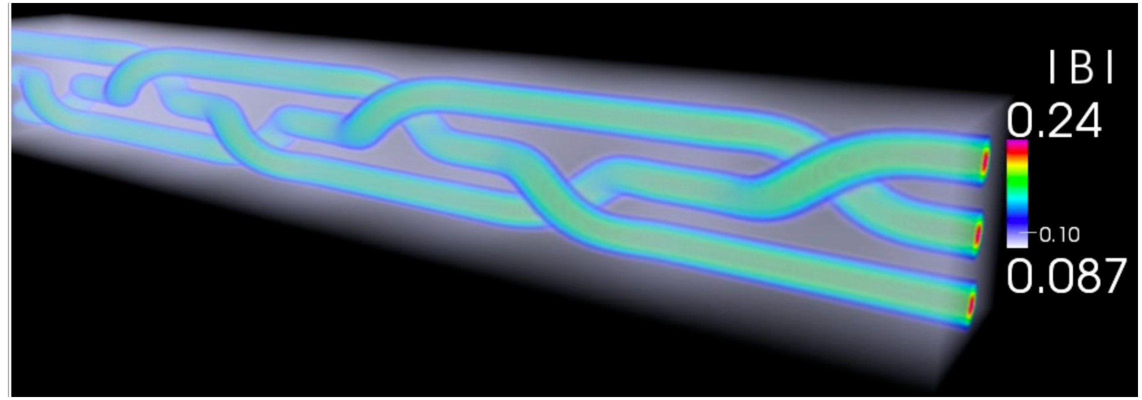


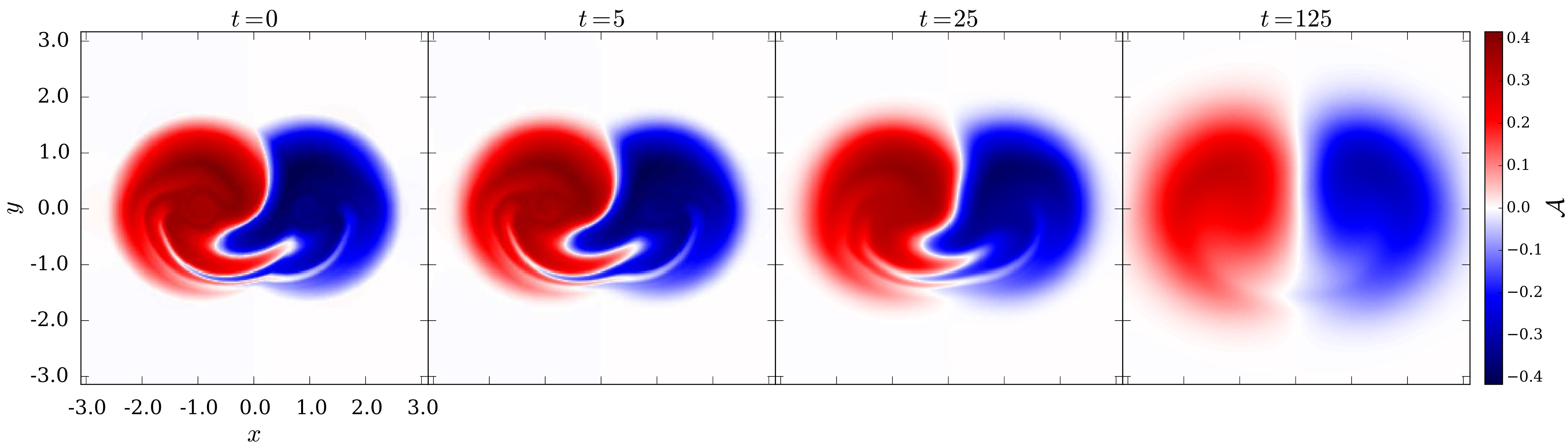
(Akhmet'ev et al. 2017)



Field Line Helicity

$$\mathcal{A}(x, y) = \int_{L(x,y)} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$





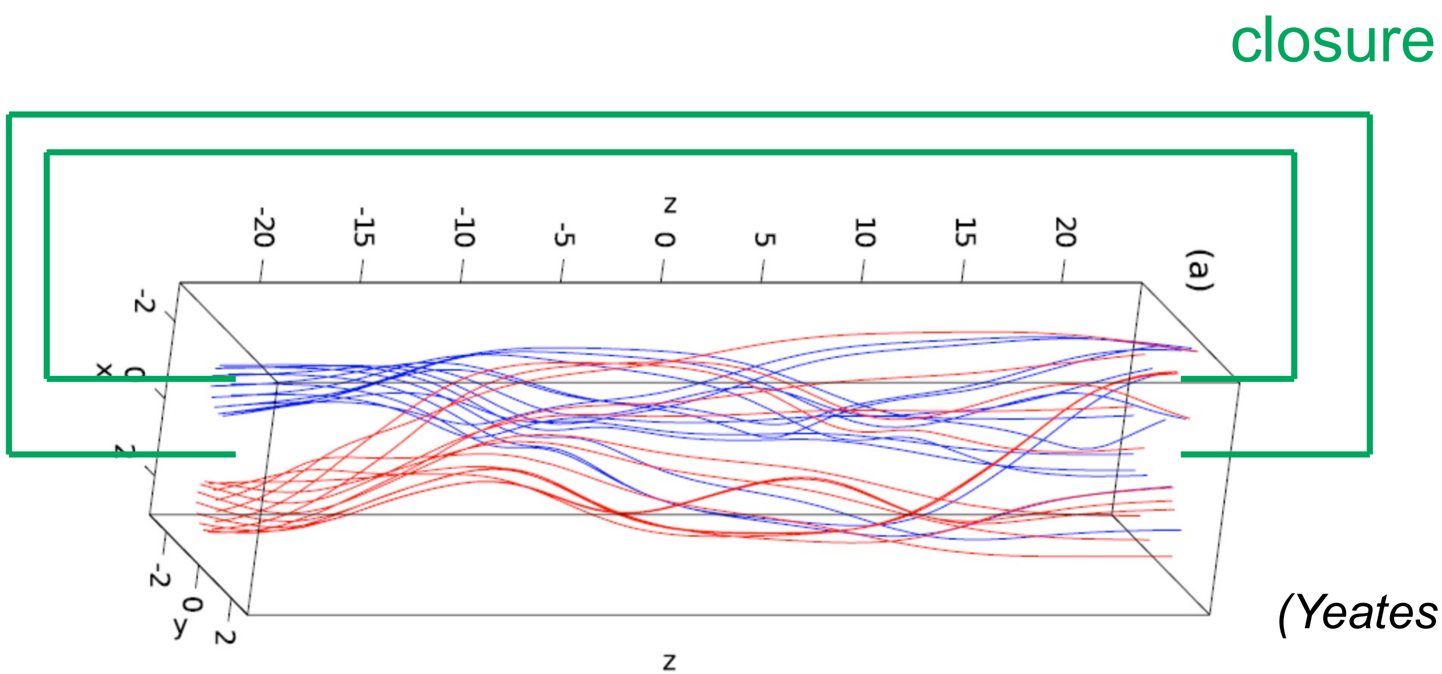
- In ideal conditions field line helicity is only being transported.

Beyond Magnetic Helicity

Describe knots, braids and links using knot polynomials:

Jones polynomials for the trefoil knot:

$$q - 1 + q^{-1}$$

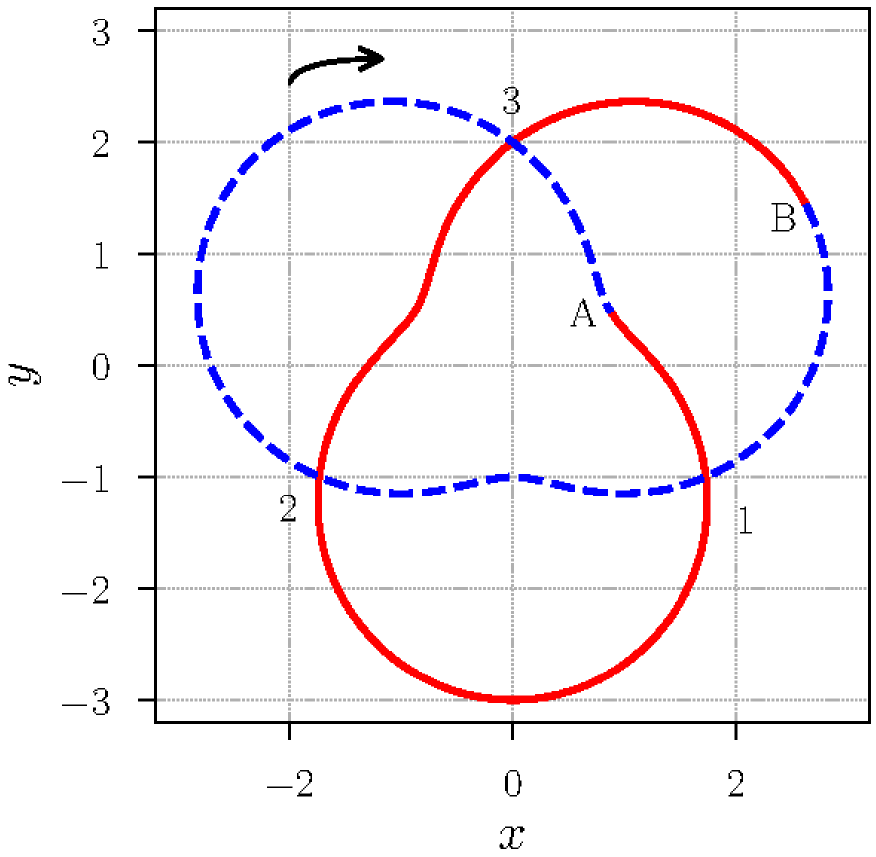


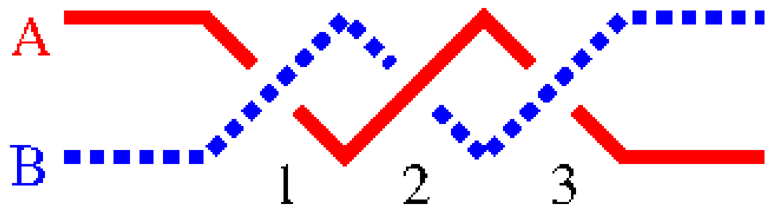
(Yeates 2011)

Use Python package Topoly to find polynomials.

(Dabrowski-Tumanski et al. (2020))

Knots and Links as Braids

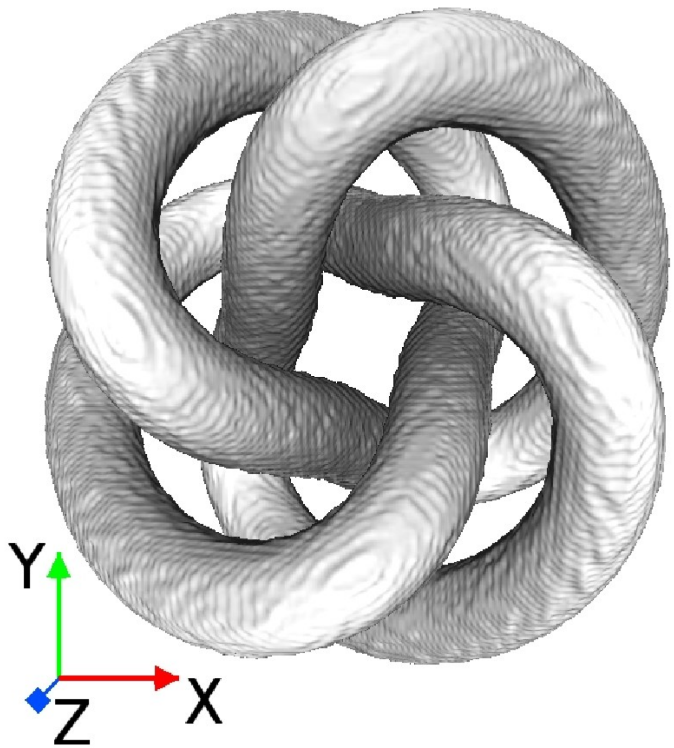




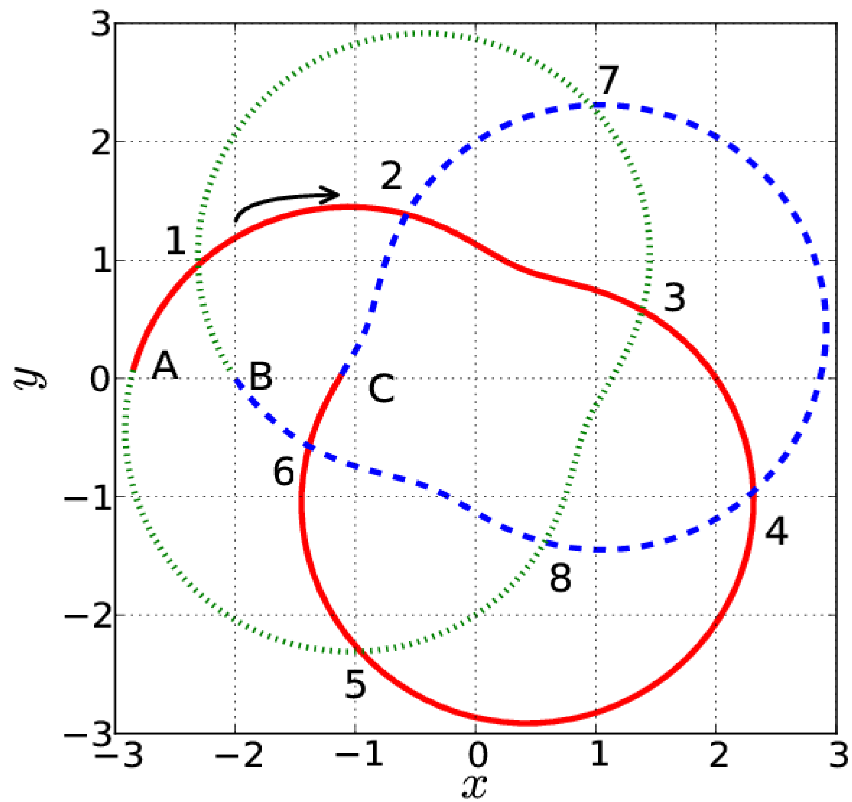
Periodic boundaries.

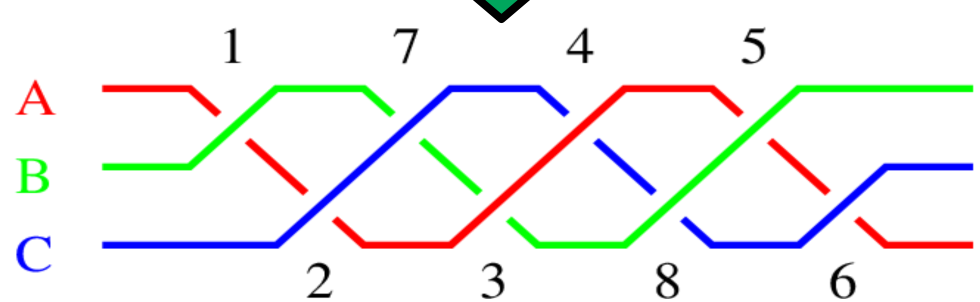
Knots and Links as Braids

need $B_z > 0$ ➡ braid representation of knots and links



4-foil knot

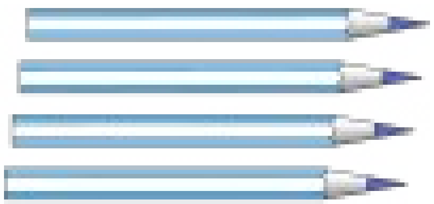




MHD Simulations

- initial condition: braids
- isothermal compressible gas
- viscous medium
- periodic in z

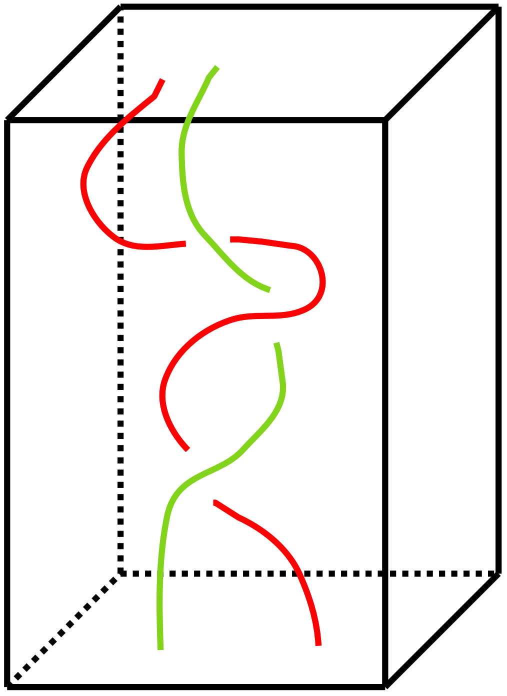
Pencil Code



$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \eta \nabla^2 \mathbf{A}$$

$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \nabla \ln \rho + \mathbf{J} \times \mathbf{B}/\rho + \mathbf{F}_{\mathrm{visc}}$$

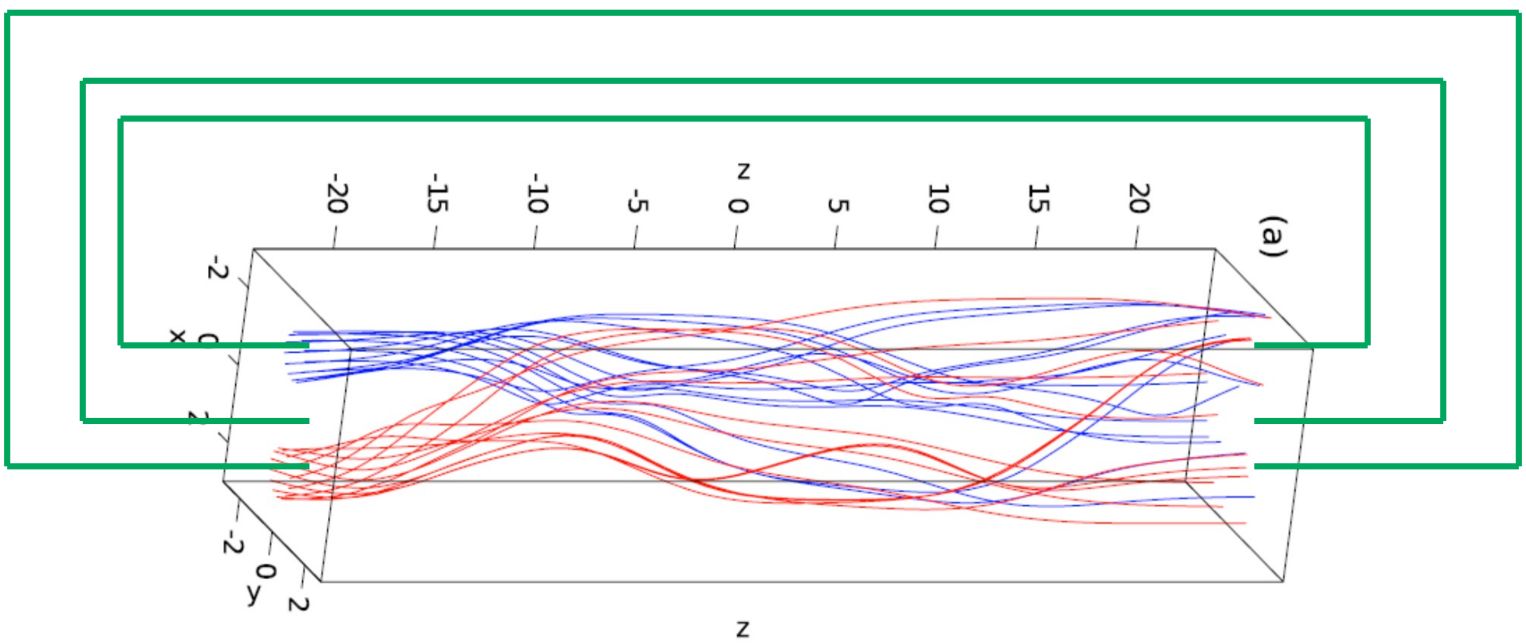
$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\nabla \cdot \mathbf{U}$$



$B_z > 0$

Link Spectrum

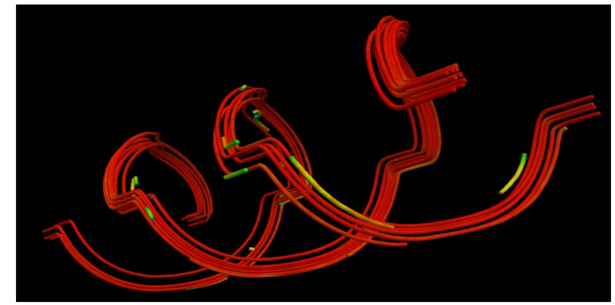
Pick a few random field lines and determine the link type.

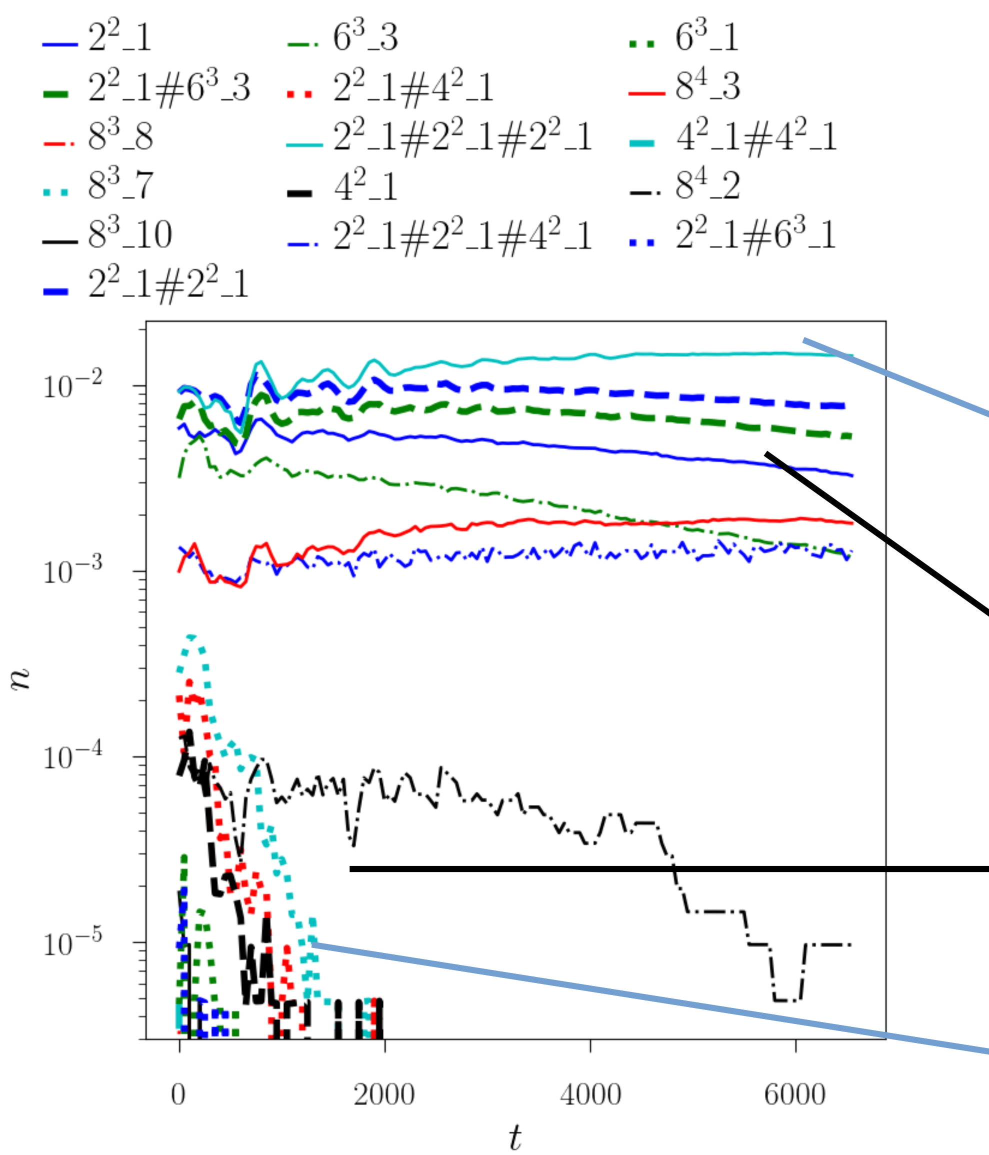


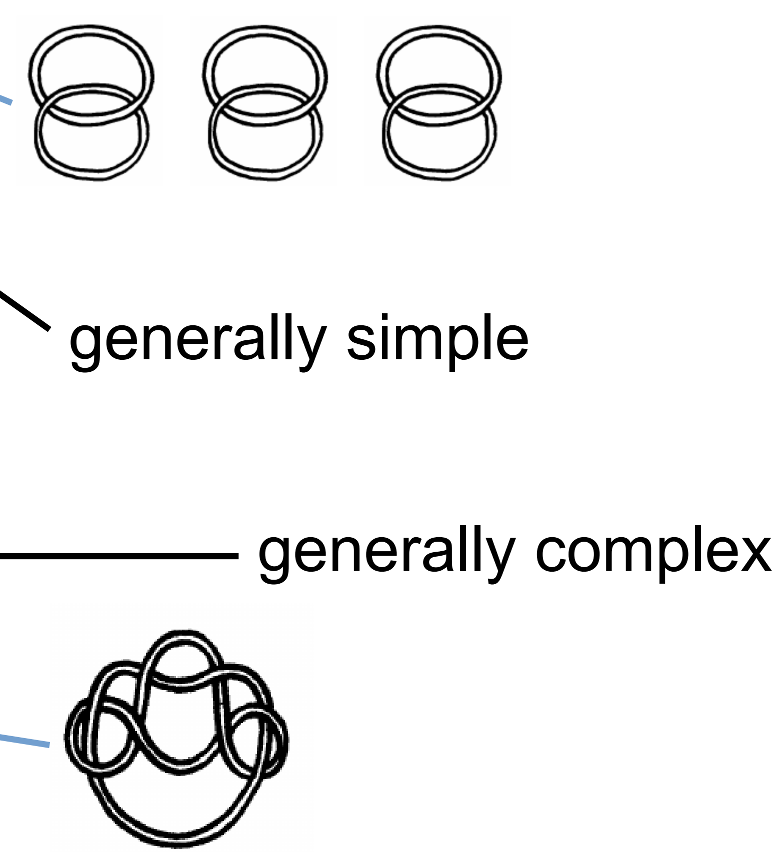
Repeat ca. 320,000 times for each snapshot.

➡ Time dependent spectrum of links.

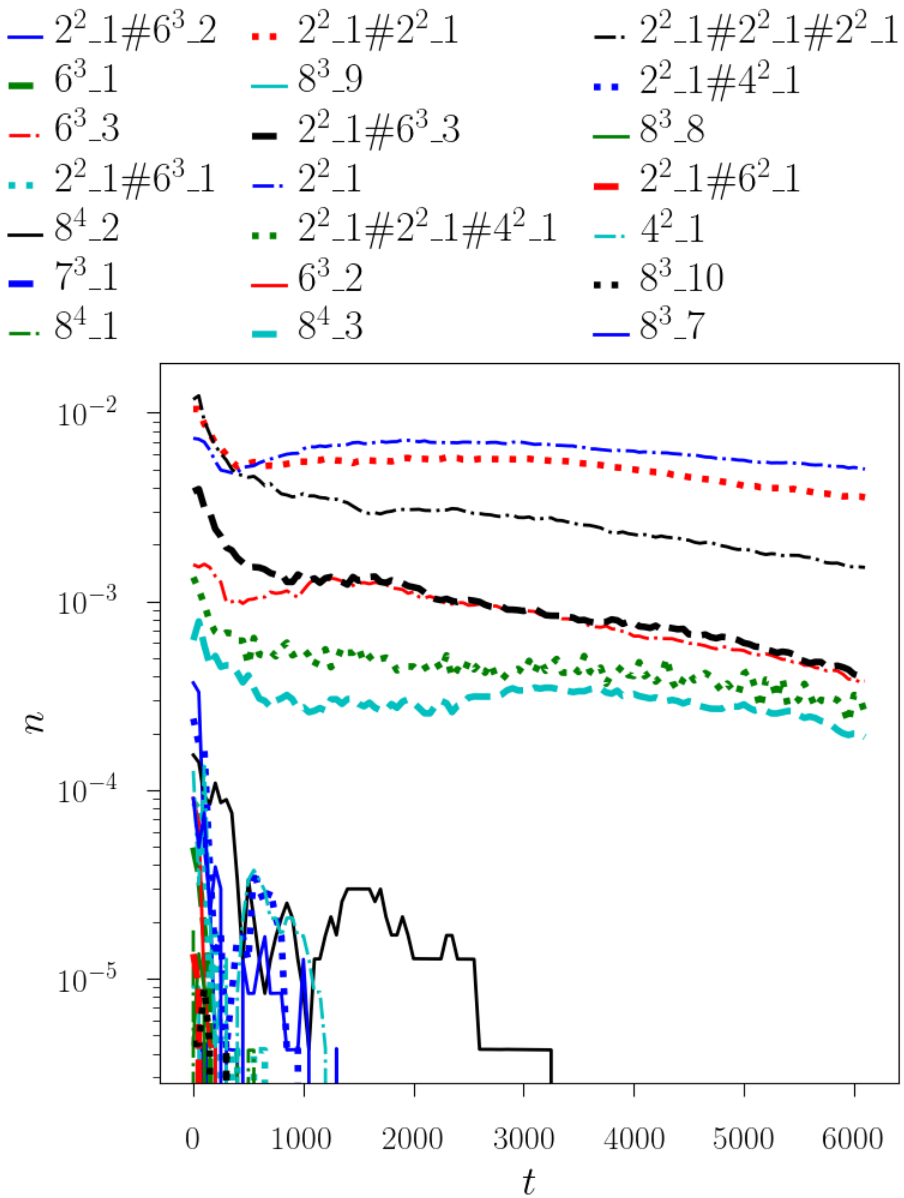
Trefoil Knot

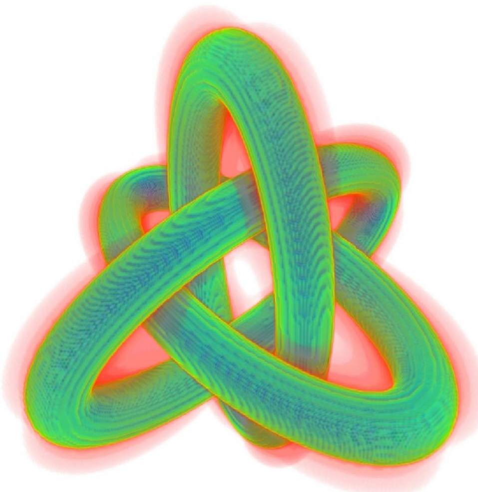


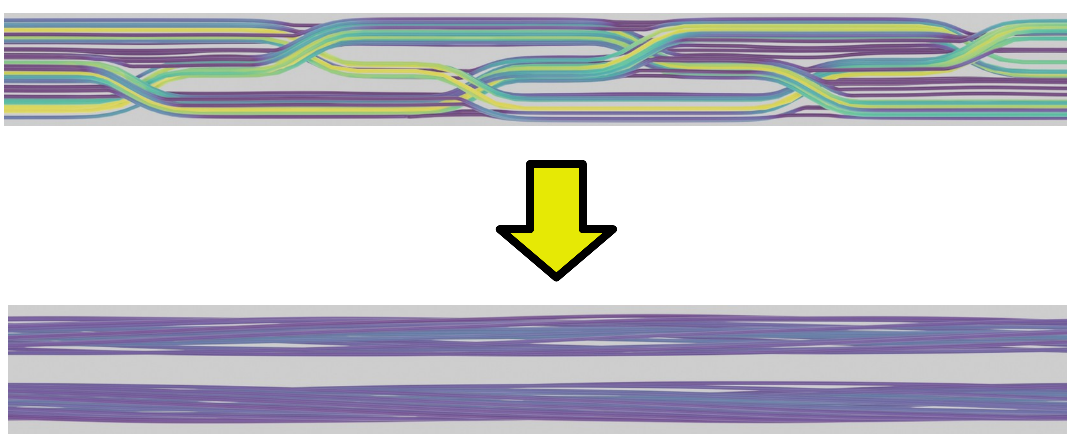




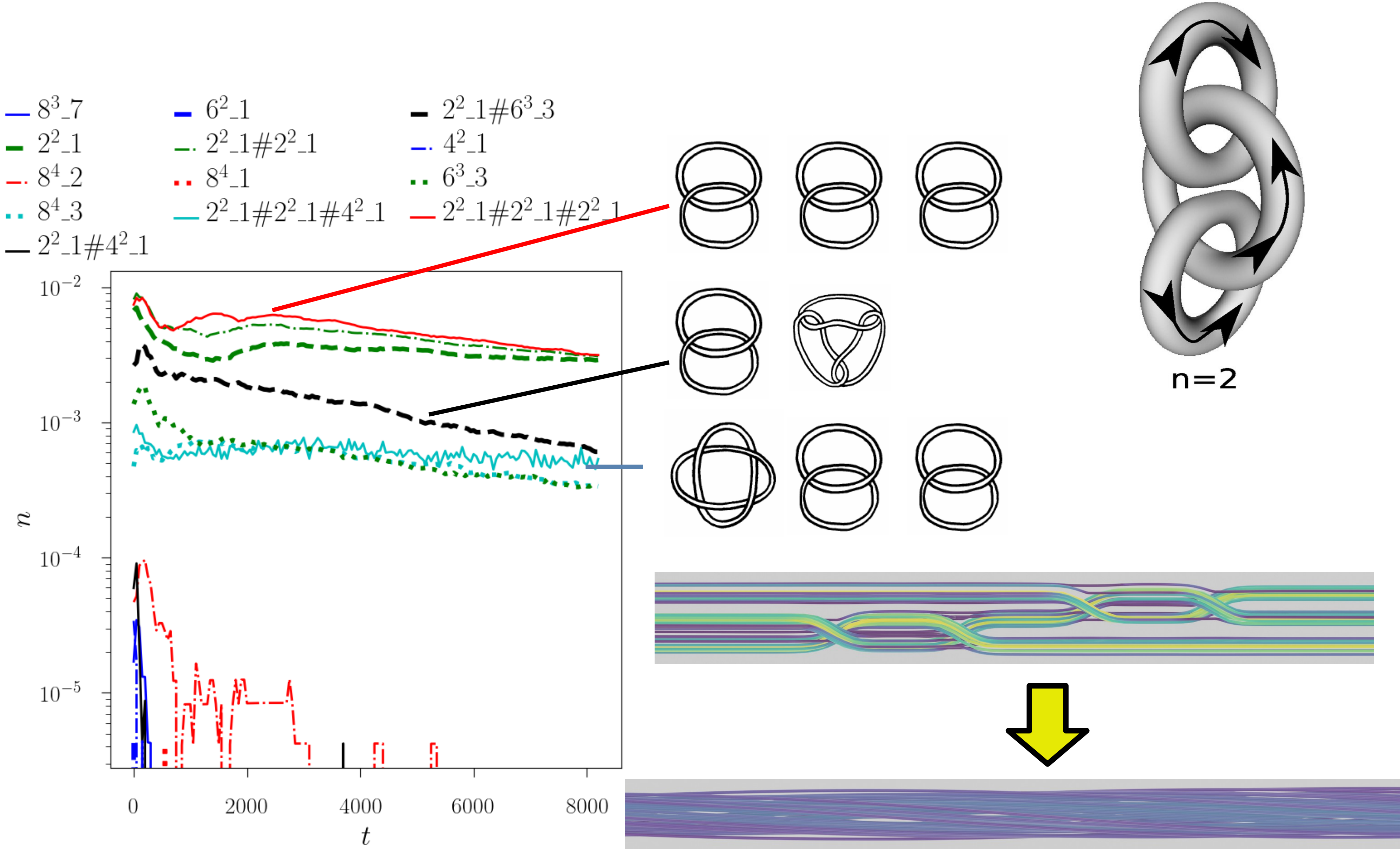
Borromean Rings



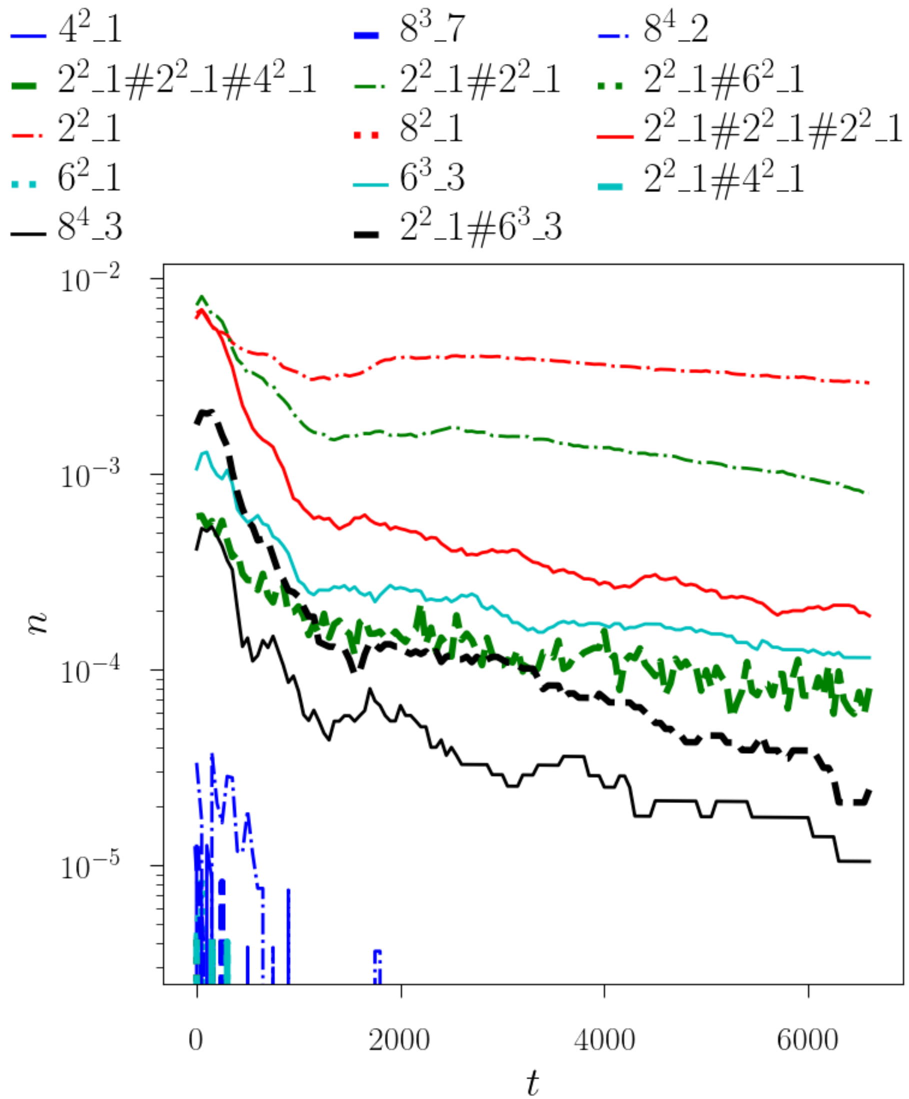


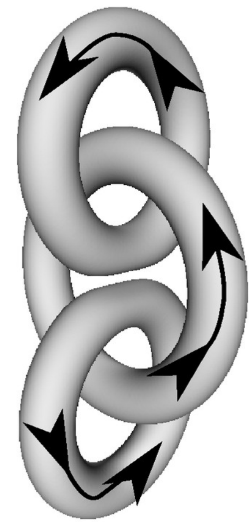


Helical 3 Rings

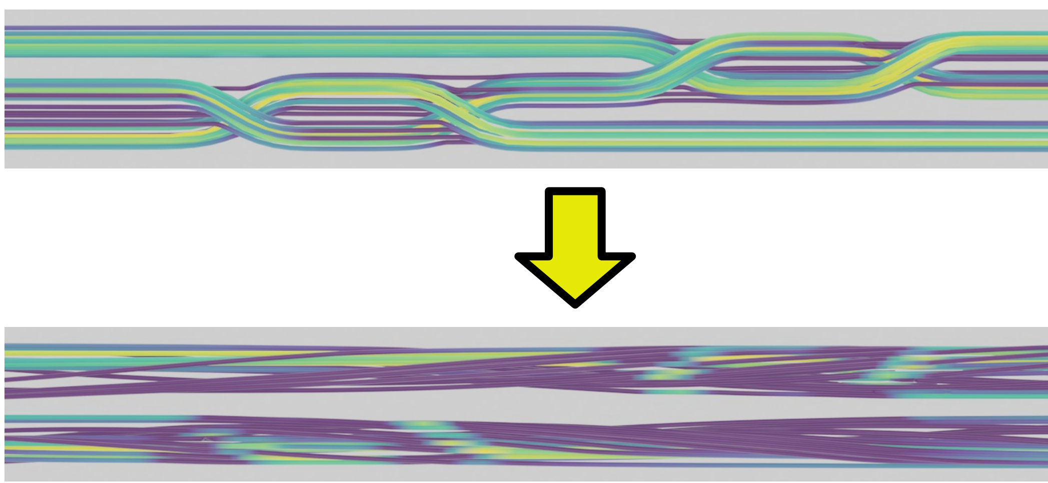


Non-Helical 3 Rings

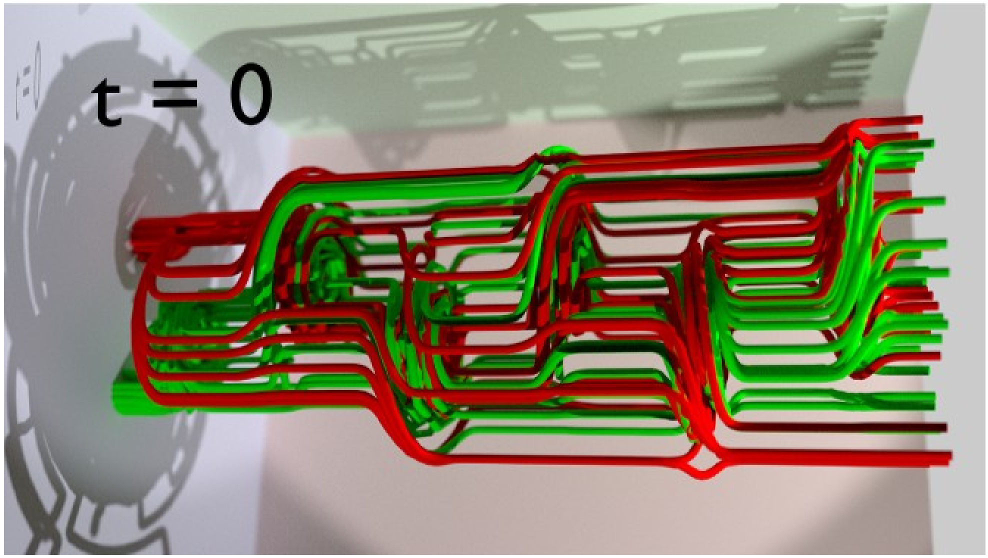




n=0



Vortex Reconnection



Field lines untangle into two twisted vortex tubes.

Conclusions

- Magnetic helicity restricts the field's dynamics.
- Twist can induce a significant helicity production.
- Quadratic helicities are ideal invariants, but require field line tracing.
- Field line helicity ideal for fields with dominant field direction.
- Knot invariants to compute spectra of braids.

simon.candelaresi@gmail.com