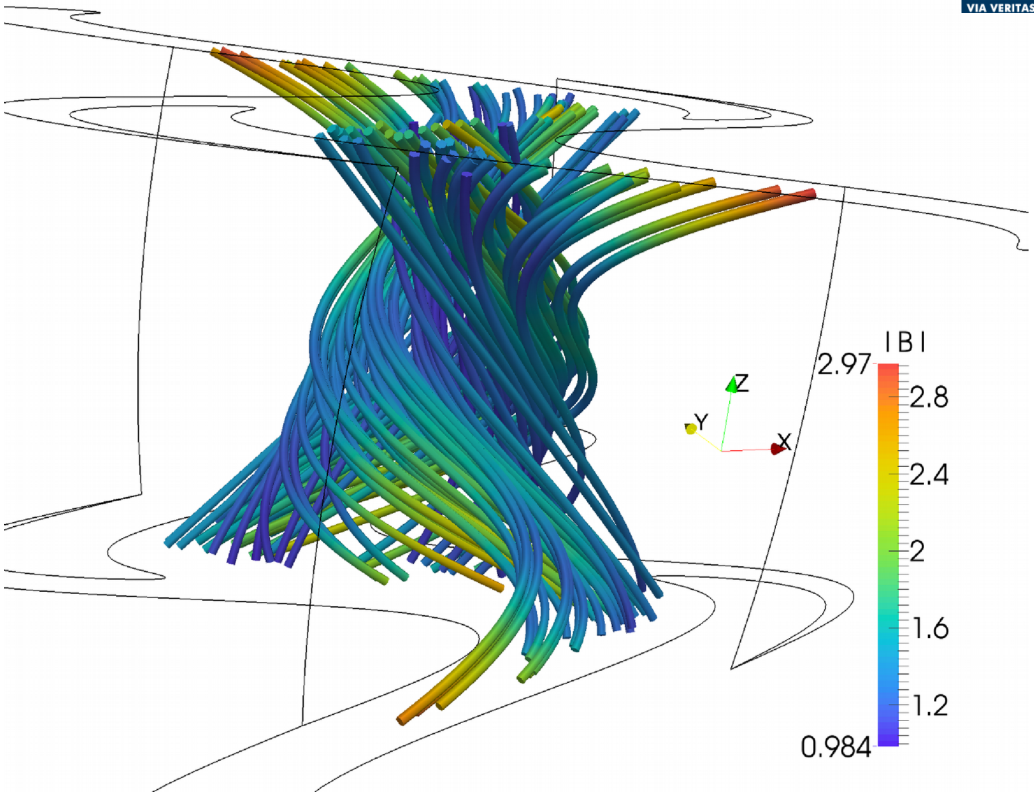
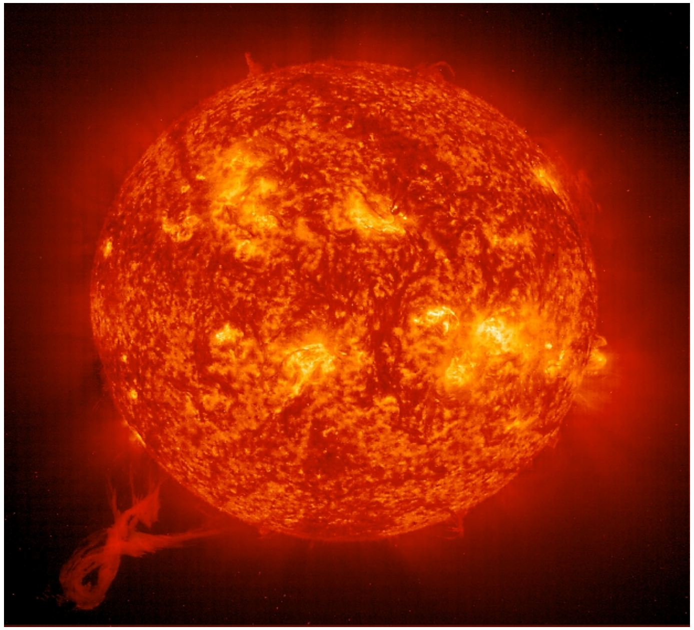
Stabilising Effect of Magnetic Field Line Topology in Plasmas

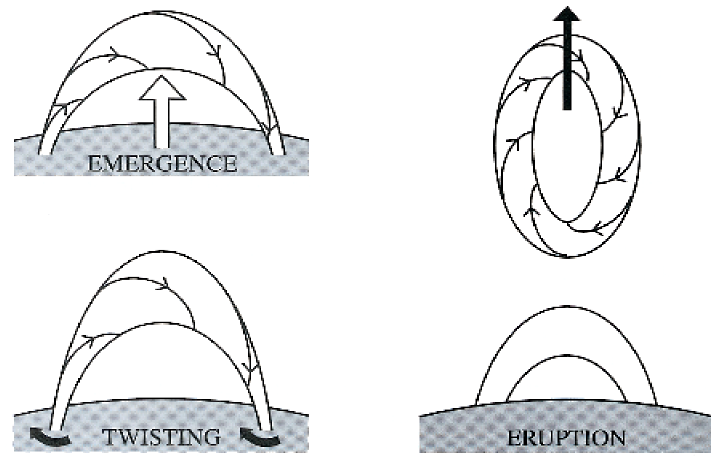
Simon Candelaresi

University of Glasgow

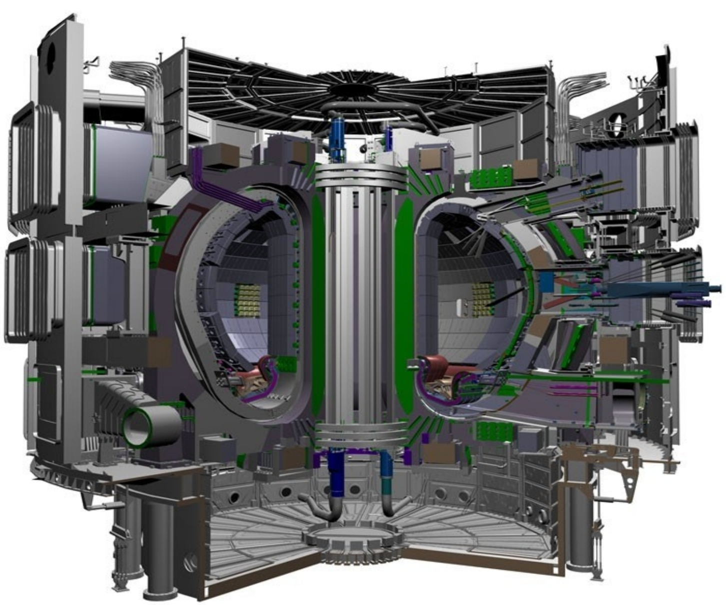


Twisted Magnetic Fields



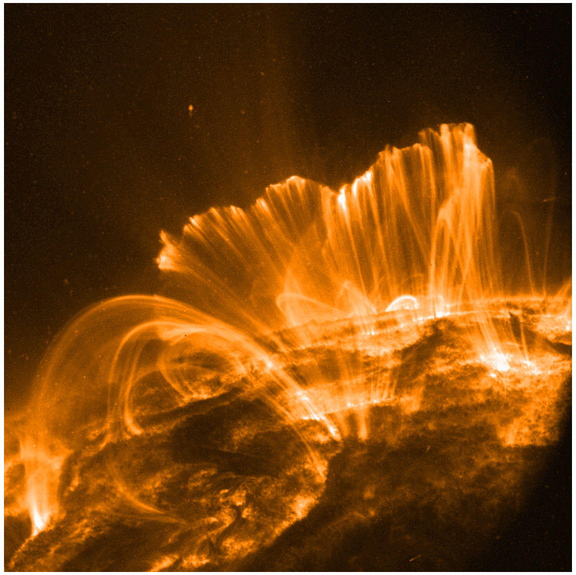


- Twisted fields are more likely to erupt *(Canfield et al. 1999)*.

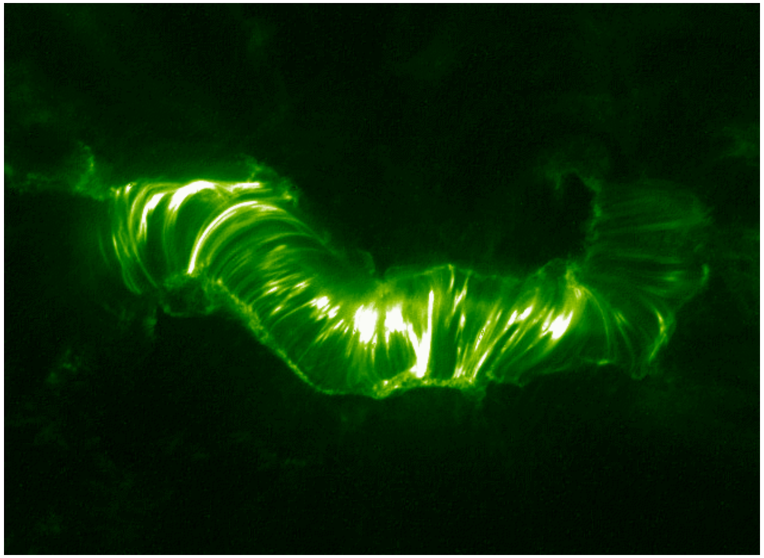


- Twist increases the stability of magnetic fields in tokamaks.

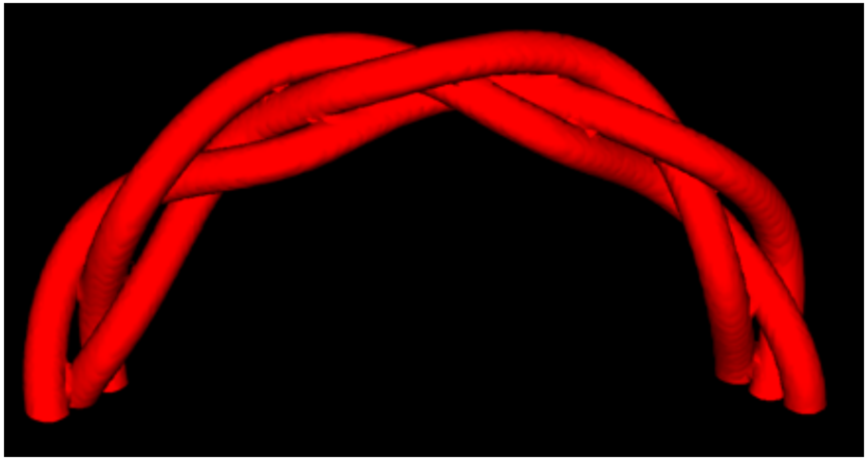
Solar Magnetic Field



(Trace)



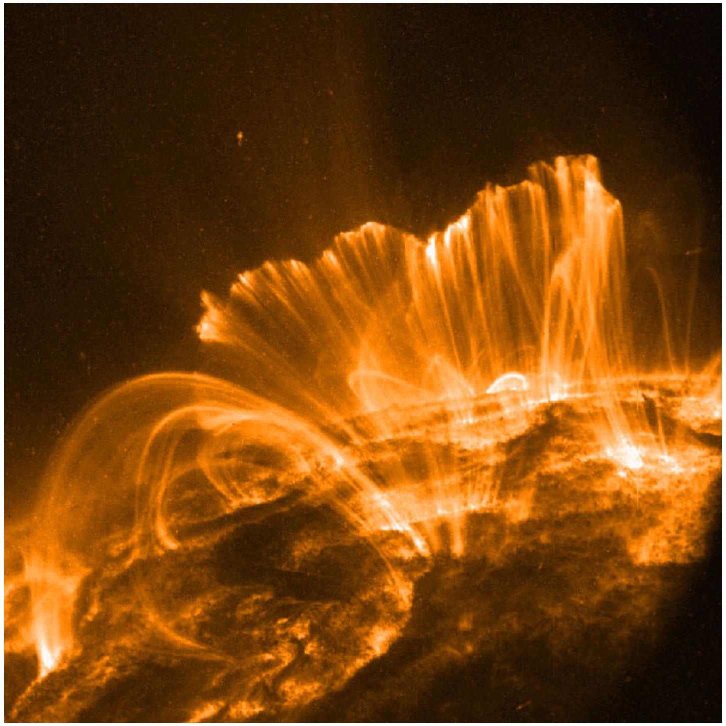
(Trace)

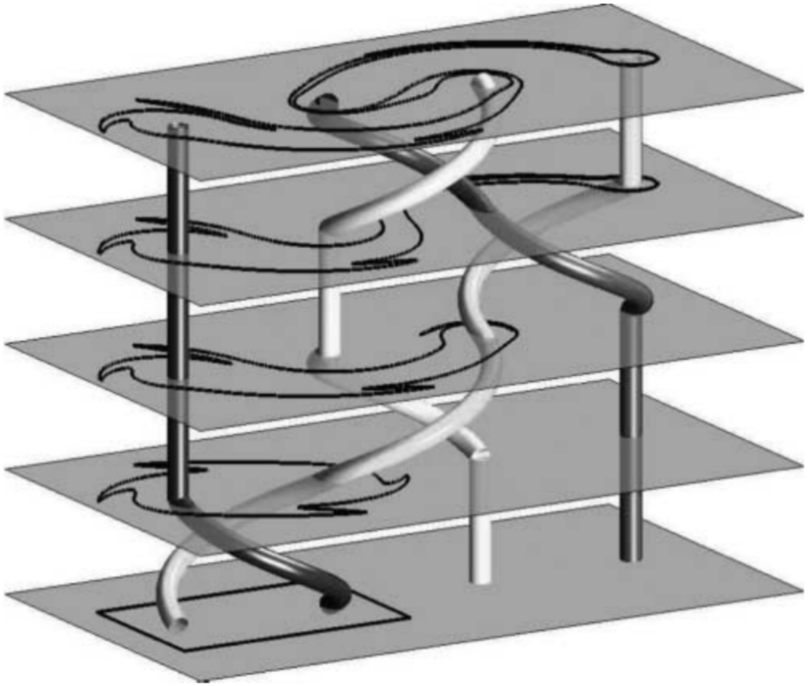


- Twisted flux tubes may rise to the corona. *(Prior and MacTaggart 2016)*.

Coronal Magnetic Fields

NASA





(Thiffeault et al. 2006)

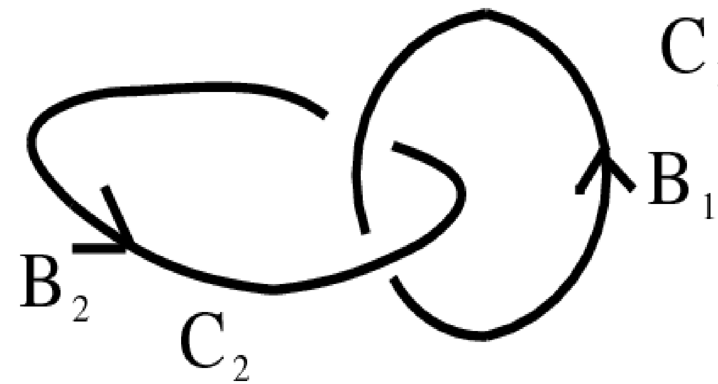
- Field line tangling in solar magnetic fields.
- Study the tangling of solar magnetic field lines.

Magnetic Helicity

Measure for the topology:

$$H_{\mathrm{M}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 2n\phi_1\phi_2$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B} \qquad \phi_i = \int_{S_i} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$



$n =$ number of mutual linking

Conservation of magnetic helicity:

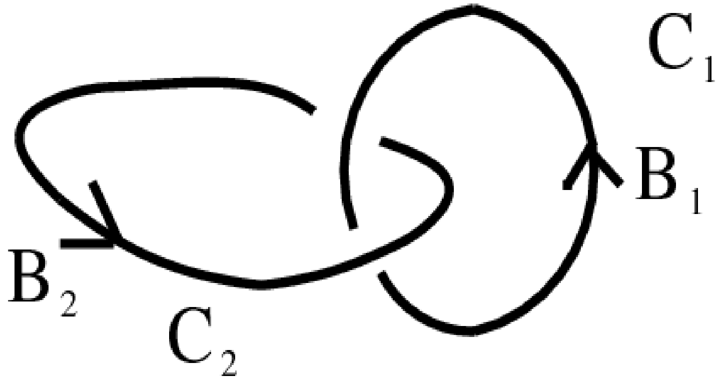
$$\lim_{\eta \to 0} \frac{\partial}{\partial t} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = 0 \qquad \eta = \text{ magnetic resistivity}$$

Realizability condition:

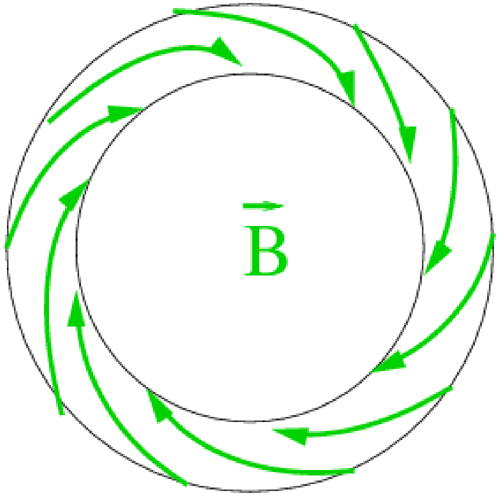
$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

Magnetic energy is bound from below by magnetic helicity.

Topologies of Magnetic Fields

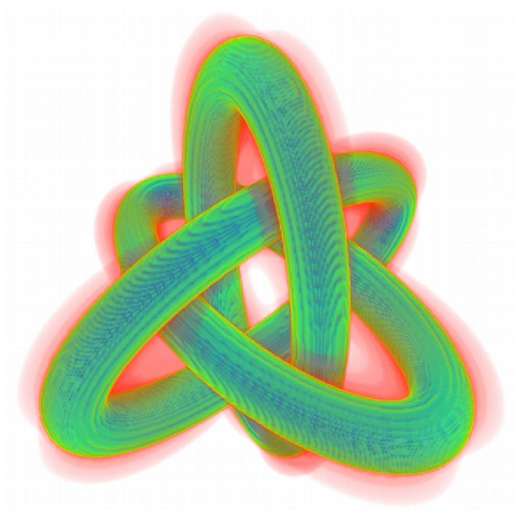


Hopf link

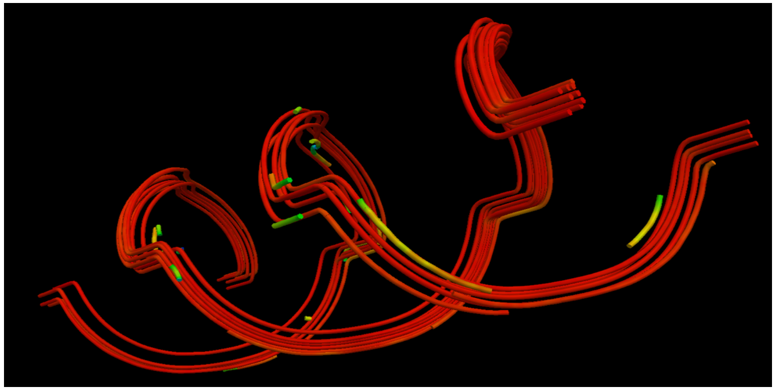


twisted field

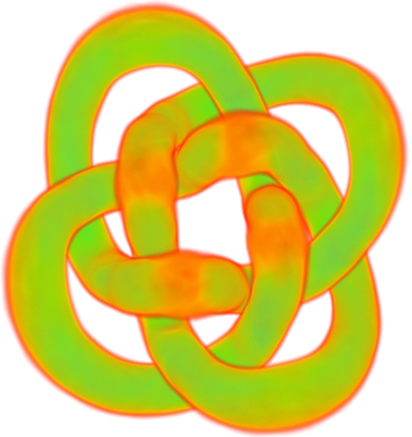
trefoil knot



Borromean rings



magnetic braid

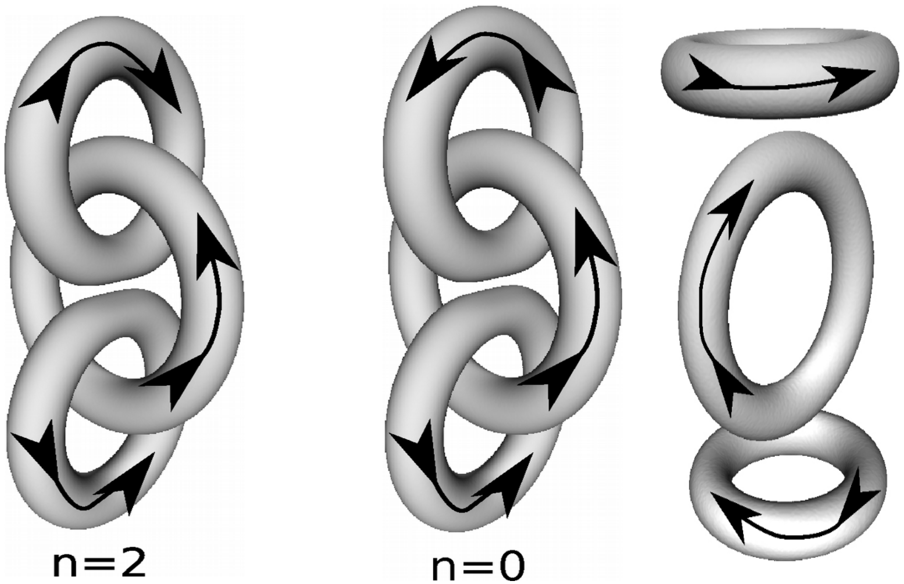


IUCAA knot

Interlocked Flux Rings

actual linking vs. magnetic helicity

$H_\mathrm{M} \neq 0$ $\qquad H_\mathrm{M} = 0$



- initial condition: flux tubes
- isothermal compressible gas
- viscous medium
- periodic boundaries

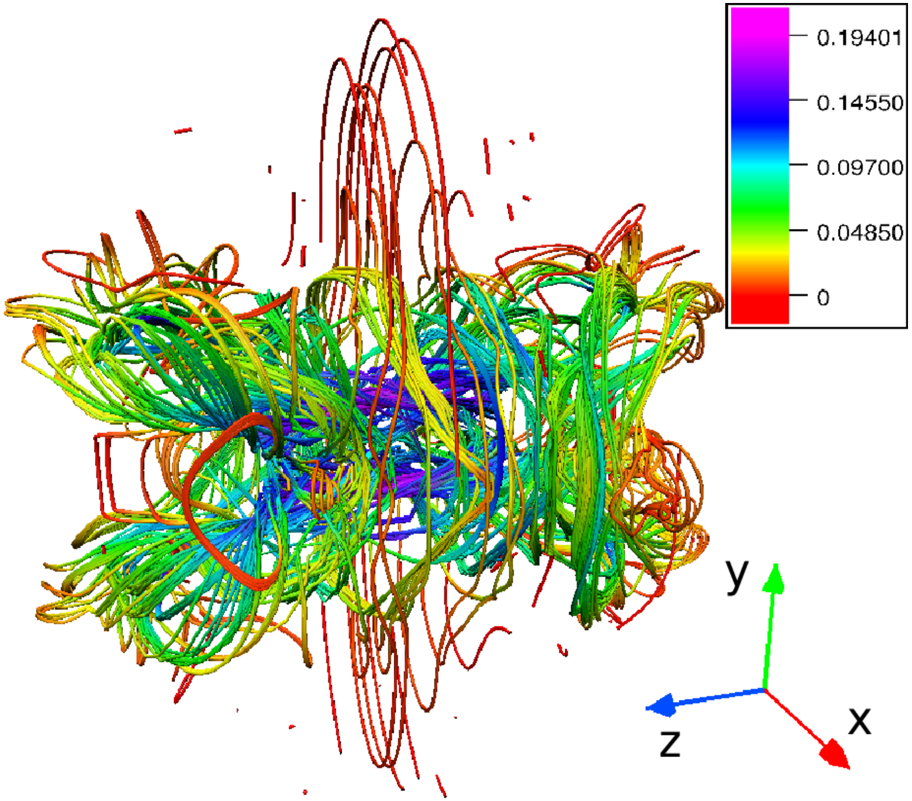
(Del Sordo et al. 2010)

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{U} \times \boldsymbol{B} + \eta \nabla^2 \boldsymbol{A} \qquad \frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \boldsymbol{U}$$

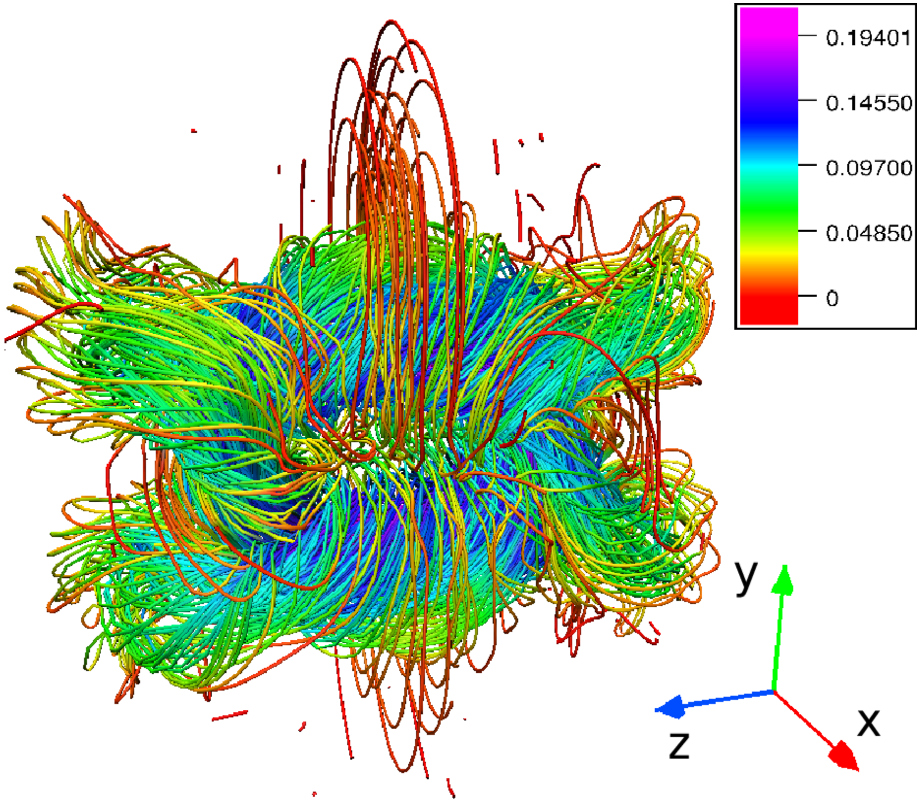
$$\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} = -c_\mathrm{S}^2 \boldsymbol{\nabla} \ln \rho + \boldsymbol{J} \times \boldsymbol{B}/\rho + \boldsymbol{F}_\mathrm{visc}$$

Interlocked Flux Rings

$\tau = 4$

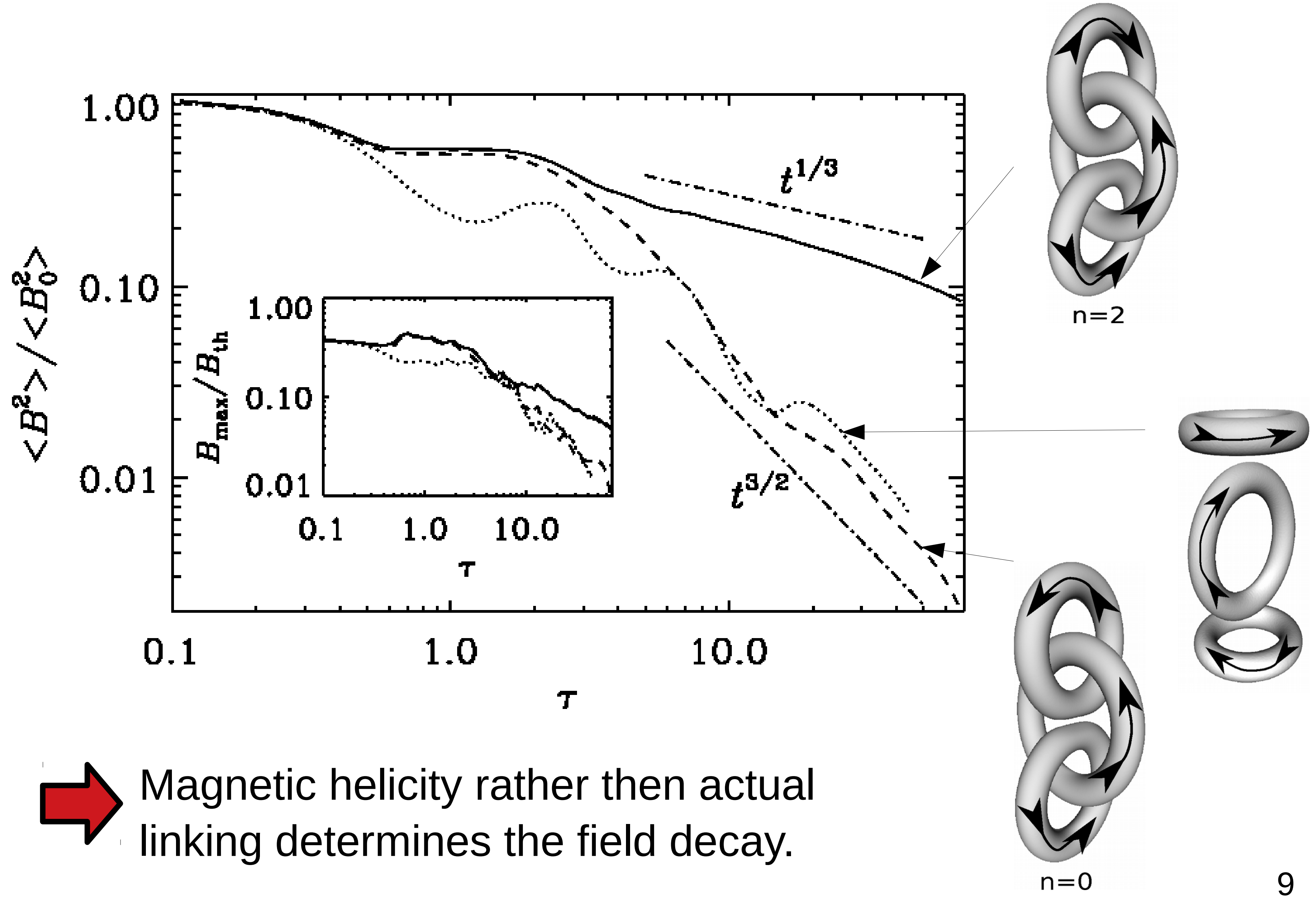


$H_{\mathrm{M}} = 0$



$H_{\mathrm{M}} \neq 0$

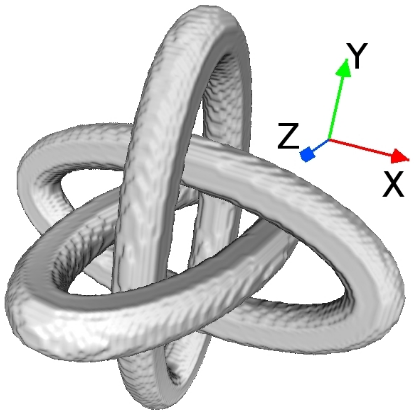
Interlocked Flux Rings



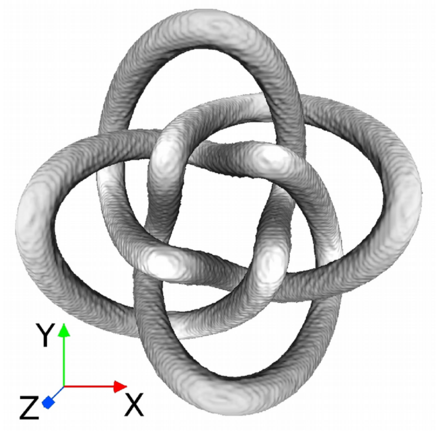
Magnetic helicity rather then actual linking determines the field decay.

IUCAA Knot and Borromean Rings

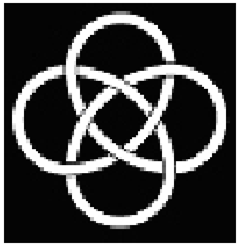
- Is magnetic helicity sufficient?
- Higher order invariants?



Borromean rings



IUCAA knot

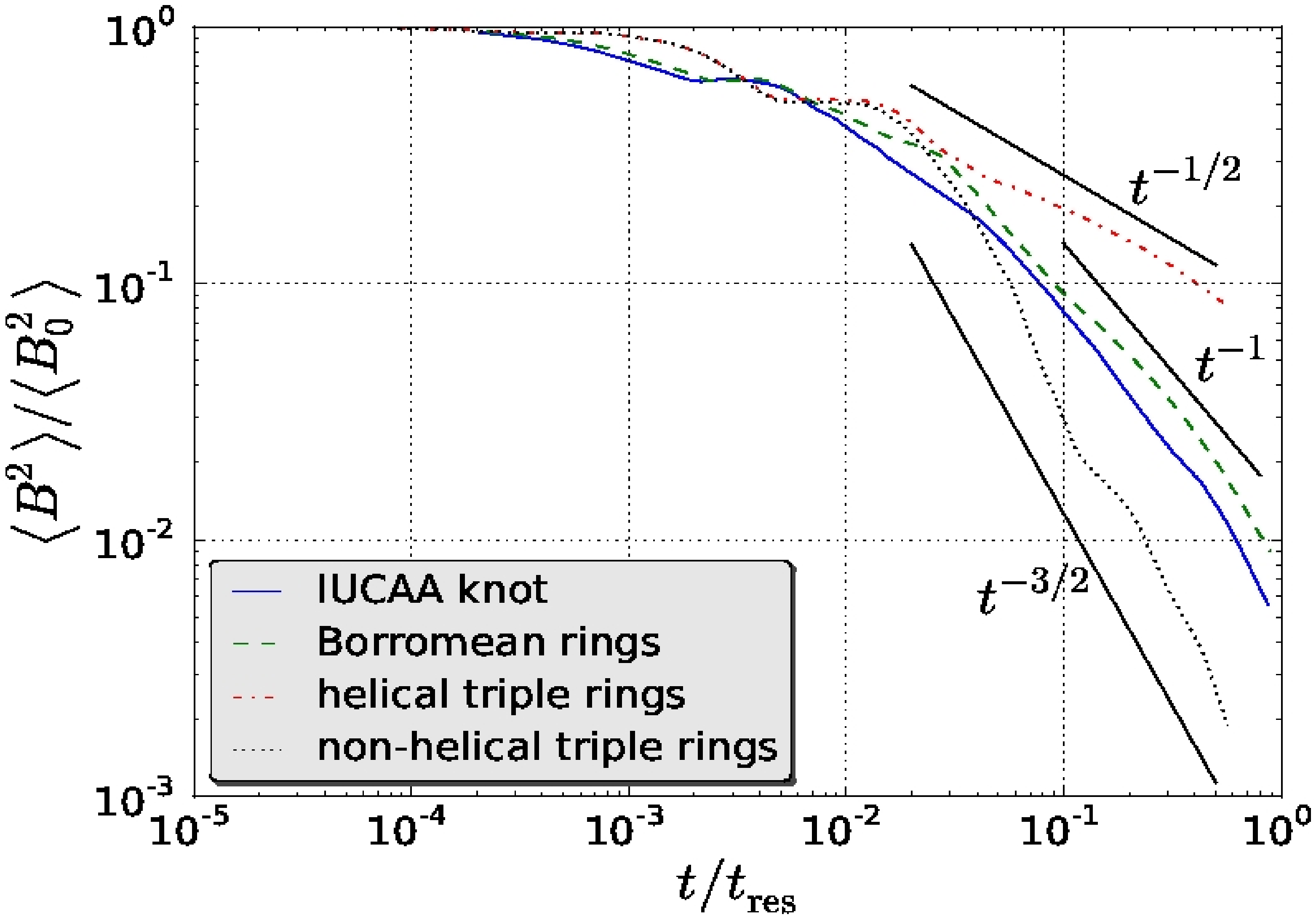


$$H_{\mathrm{M}} = 0$$

(Candelaresi and Brandenburg 2011)

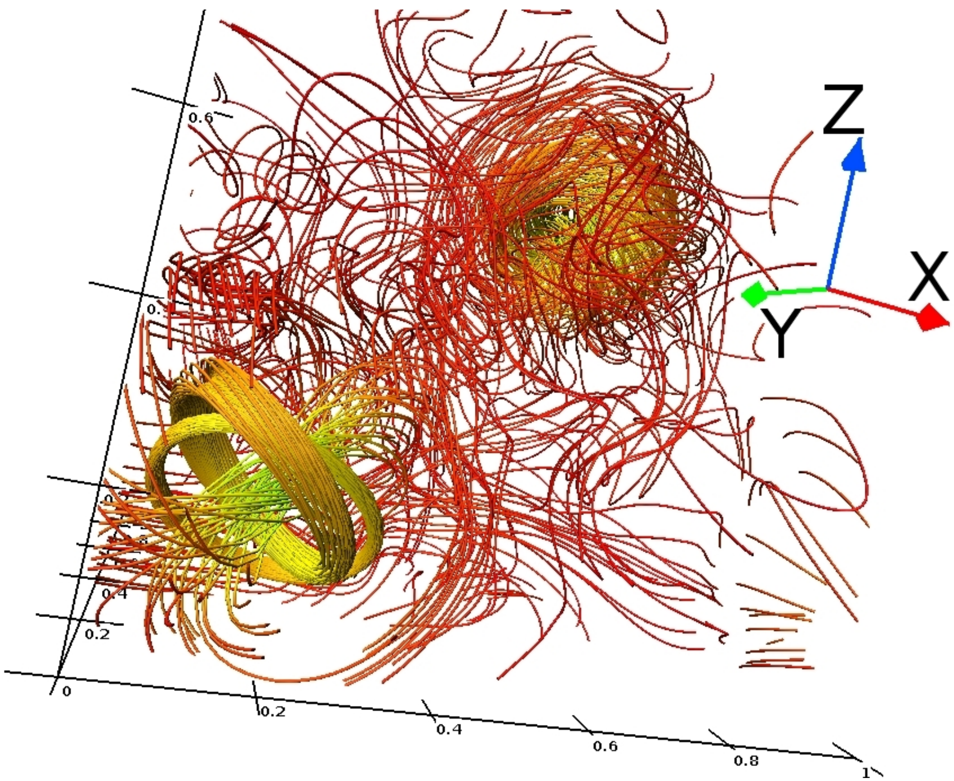
IUCAA = The Inter-University Centre for Astronomy and Astrophysics, Pune, India

Magnetic Energy Decay

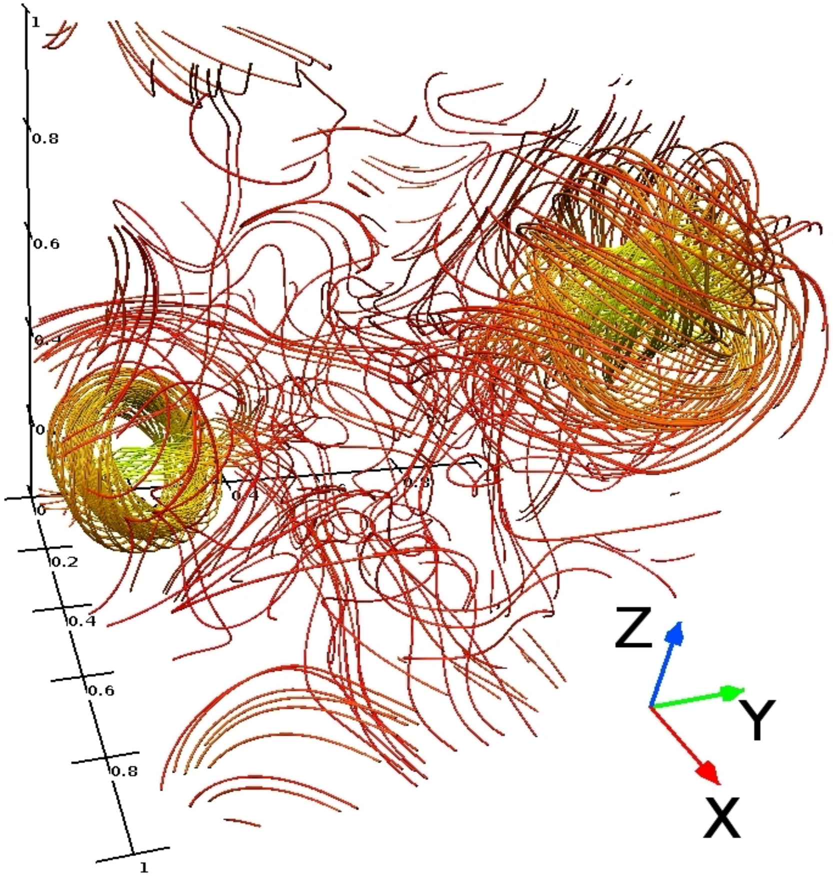


Higher order invariants?

Reconnection characteristics



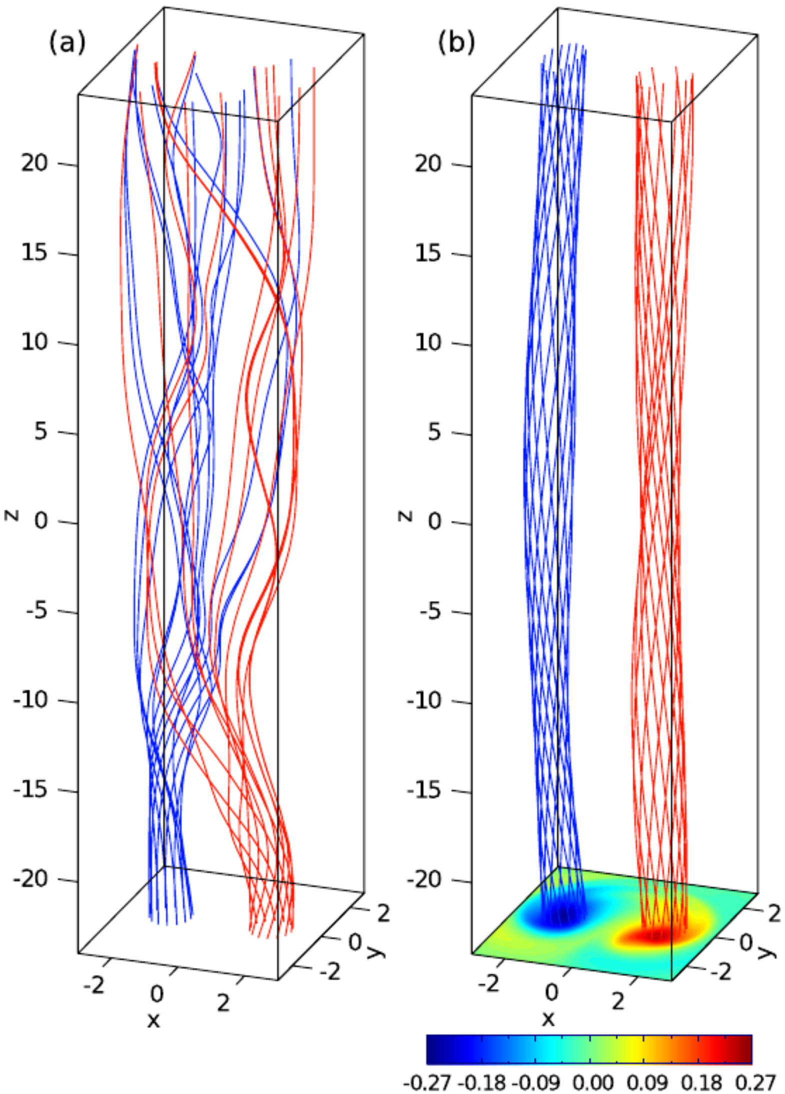
$t = 70$



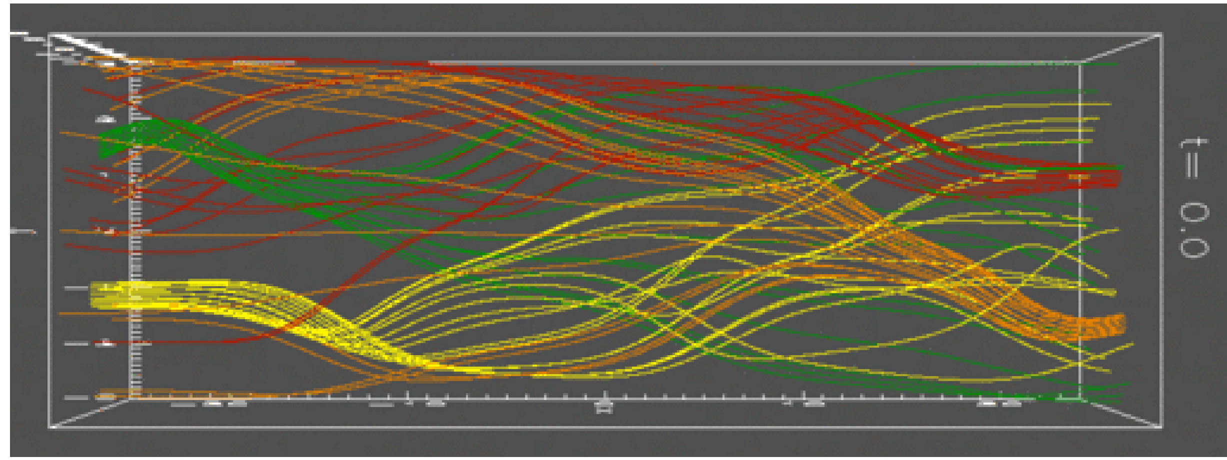
$t = 78$

3 rings ➡ Twisted ring + interlocked rings ➡ 2 twisted rings

Magnetic Braid



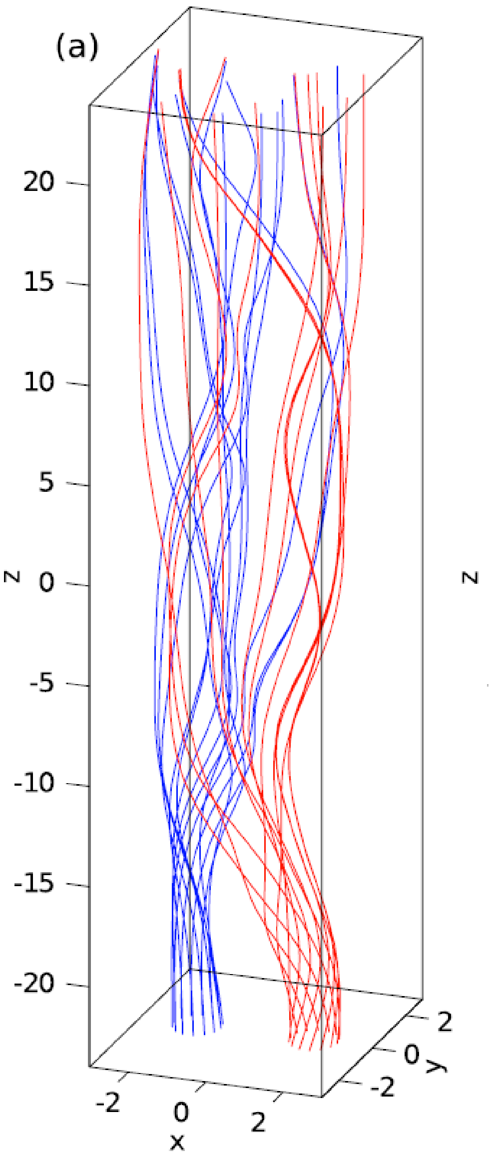
(Yeates 2011)



(Wilmot-Smith 2010)

- Periodic braid topologically equivalent to Borromean rings.
- Separation into two twisted field regions.
- Conserved invariants like fixed point index and field line helicity.

Fixed Point Index



(Yeates et al. 2011)

Trace magnetic field lines from z_0 to z.

mapping: $(x, y) \rightarrow \mathbf{F}_z(x, y)$

fixed points: $\mathbf{F}_1(x, y) = (x, y)$

Color coding:

Compare (x, y) with $\mathbf{F}_1(x, y)$:

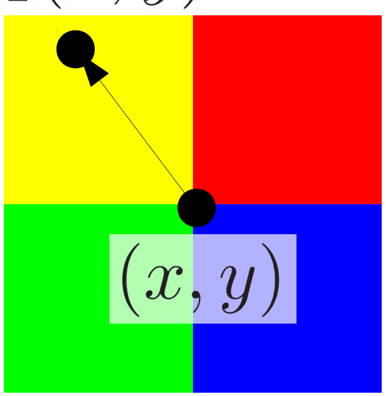
$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y > y$ ⇨ red

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y > y$ ⇨ yellow

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y < y$ ⇨ green

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y < y$ ⇨ blue

$\mathbf{F}_1(x, y)$



Stability criteria

constraint | equilibrium

Woltjer (1958): $$\frac{\partial}{\partial t}\int_V \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V = 0 \qquad \boldsymbol{\nabla}\times\mathbf{B} = \alpha\mathbf{B}$$

Taylor (1974): $$\frac{\partial}{\partial t}\int_{\tilde{V}} \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V = 0 \qquad \boldsymbol{\nabla}\times\mathbf{B} = \alpha(a,b)\mathbf{B}$$

constant along field line

V = total volume $\qquad$ $\tilde{V}$ = volume along magnetic field line

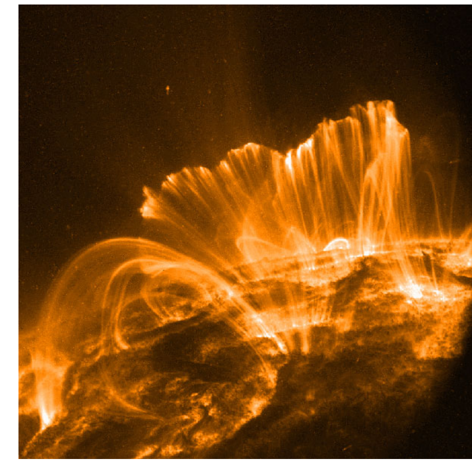
⇨ **Taylor state not reached due to fixed point conservation.**

(Yeates et al. 2011)

Force-Free Magnetic Fields

Solar corona: low plasma beta and magnetic resistivity

NASA

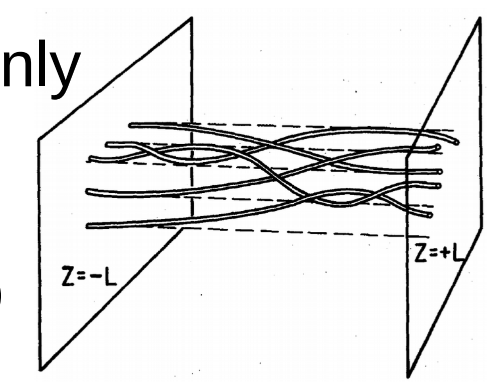


- Force-free magnetic fields
- Minimum energy state

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \;\Leftrightarrow\; \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

Parker: Equilibrium with the same topology exists only if the twist varies uniformly along the field lines. Strongly braided fields -> topological dissipation.

(Parker 1972)



Braided fields from foot point motion complex enough. *(Parker 1983)*

Solutions possible with filamentary current structures (sheets).

(Mikic 1989, Low 2010)

Methods

Ideal (non-resistive) evolution

Frozen in magnetic field ➡ use Lagrangian method

(Batchelor, 1950)

Preserves topology and divergence-freeness.

Magneto-frictional term: $\mathbf{u} = \gamma \mathbf{J} \times \mathbf{B} \qquad \mathbf{J} = \nabla \times \mathbf{B}$

➡ $\frac{\mathrm{d}E_\mathrm{M}}{\mathrm{d}t} < 0$ *(Craig and Sneyd 1986)*

Fluid with pressure: $\mathbf{u} = \mathbf{J} \times \mathbf{B} - \beta \nabla \rho$

Fluid with inertia: $\mathrm{d}\mathbf{u}/\mathrm{d}t = (\mathbf{J} \times \mathbf{B} - \nu \mathbf{u} - \beta \nabla \rho)/\rho$

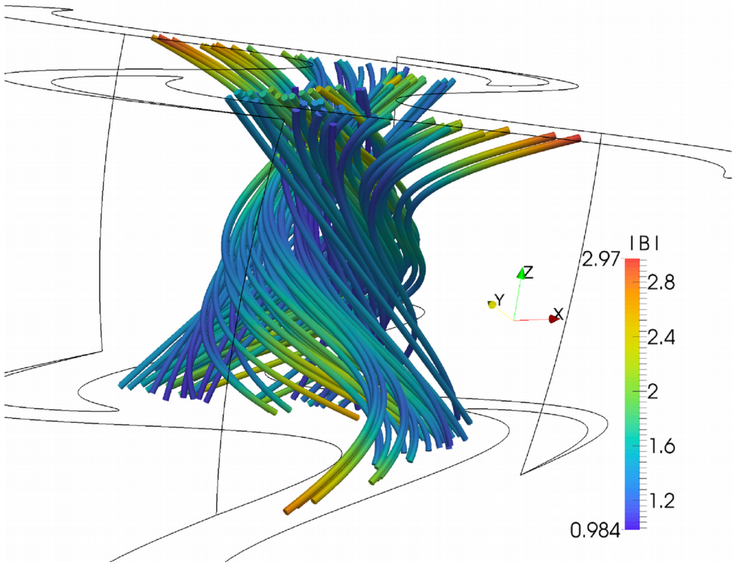
For $\mathbf{J} = \nabla \times \mathbf{B}$ use mimetic numerical operators.

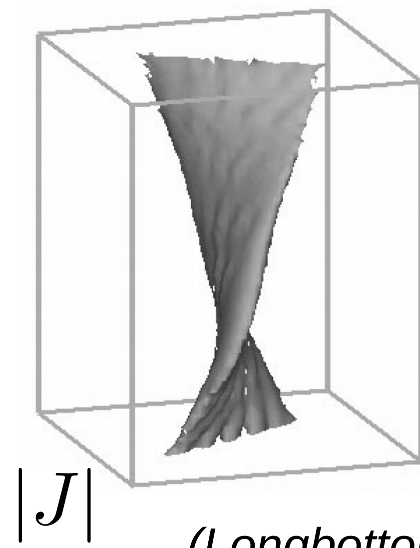
(Hyman, Shashkov 1997)

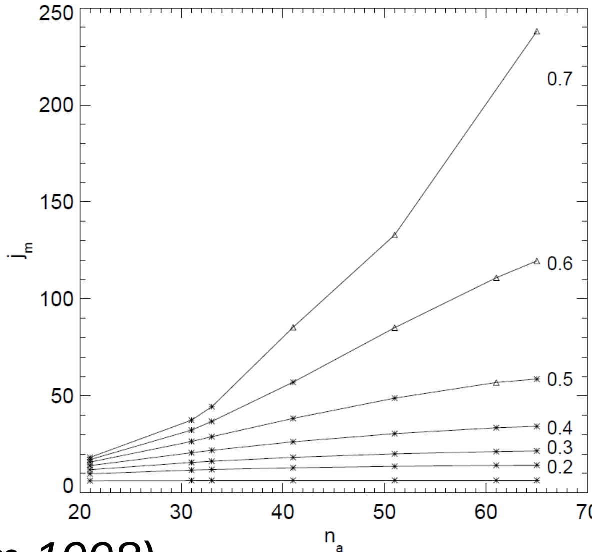
Own GPU code GLEMuR: (https://github.com/SimonCan/glemur).

(Candelaresi et al. 2014)

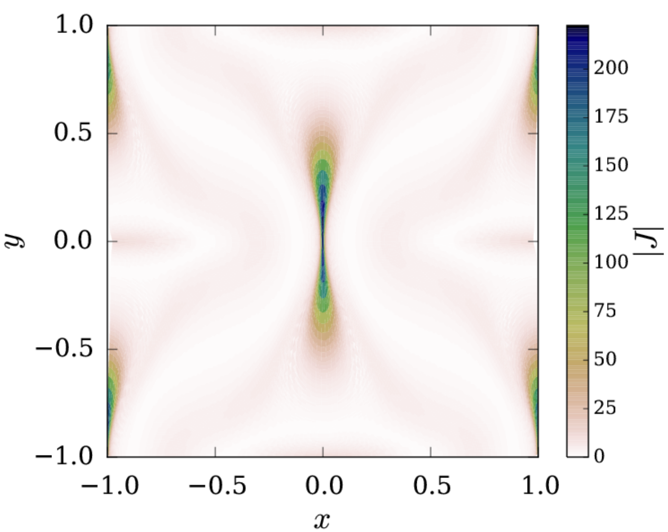
Distorted Magnetic Fields

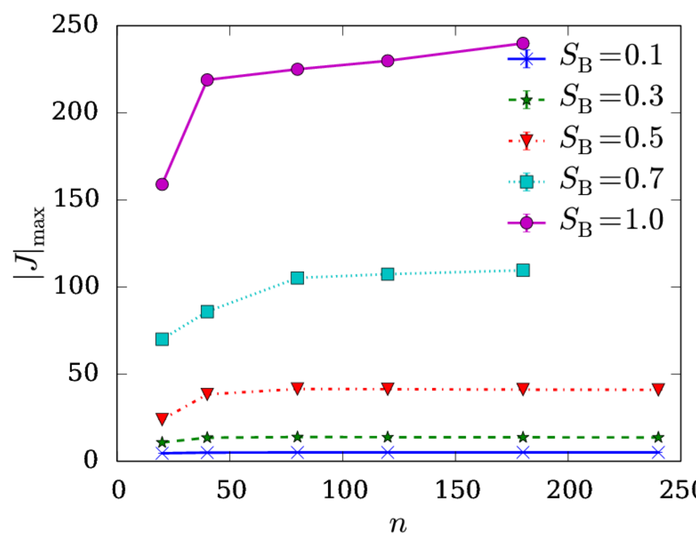






(Longbottom 1998)

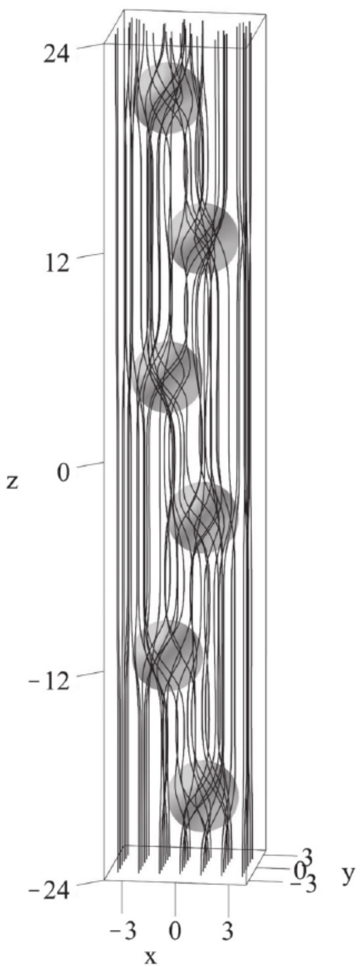


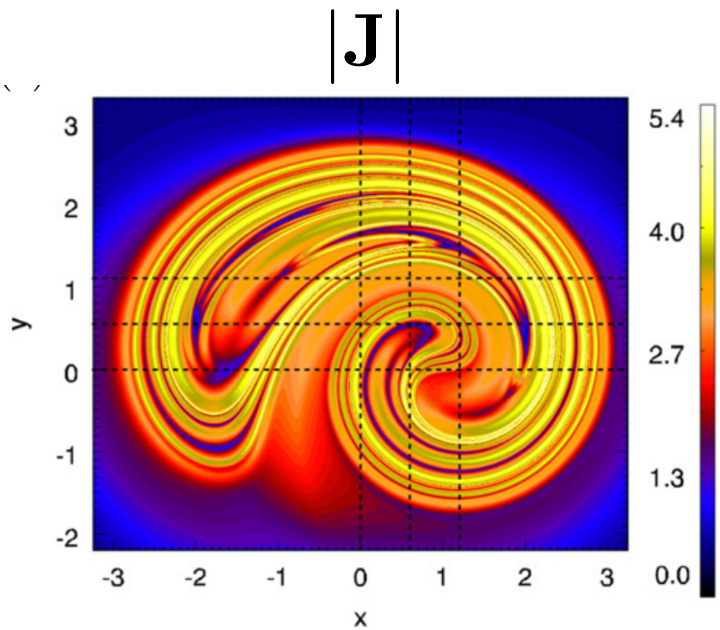


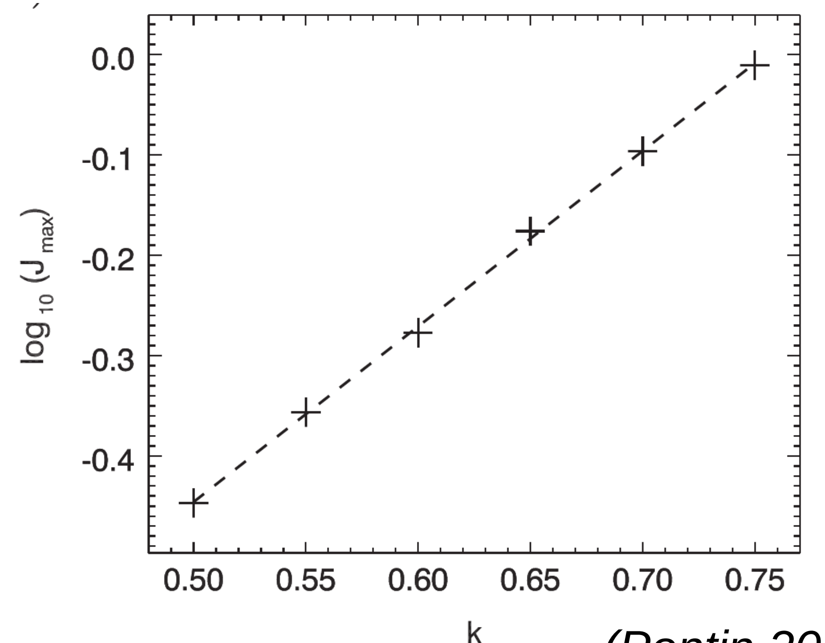
- resolved current concentrations
- shear leads to strong currents

(Candelaresi et al. 2015)

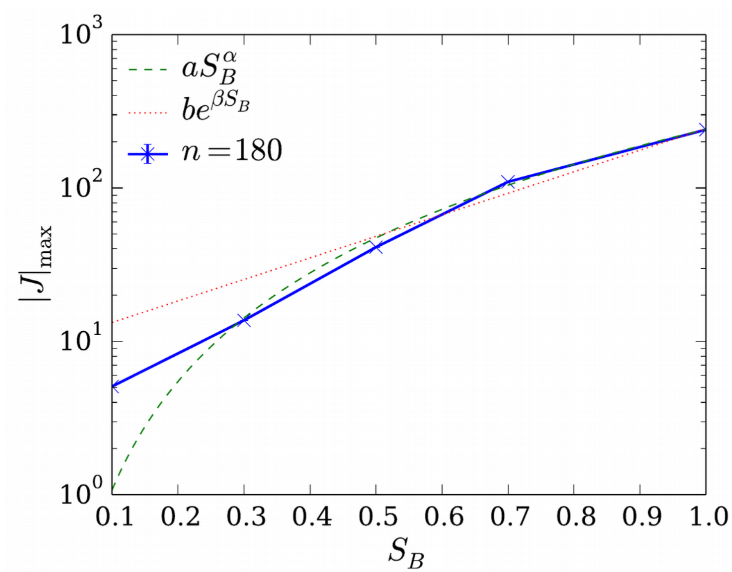
Exponential Increase in Current







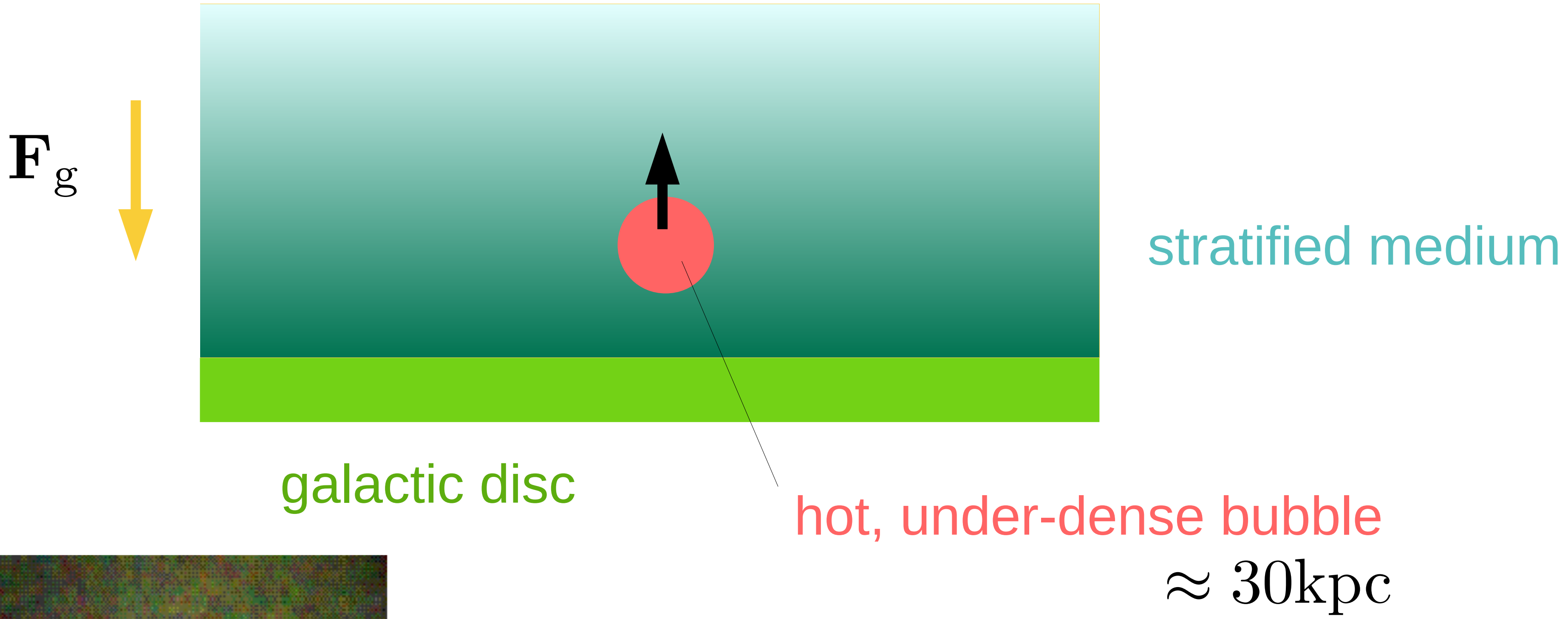
(Pontin 2015)

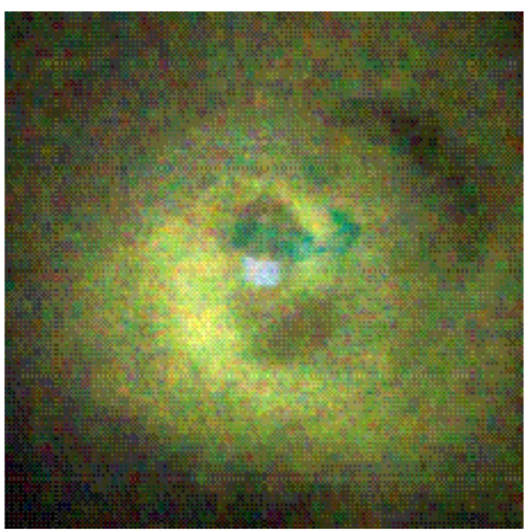


(Candelaresi 2015)

- lengths decrease exp. with complexity
- current increases exp. with complexity

Intergalactic Bubbles

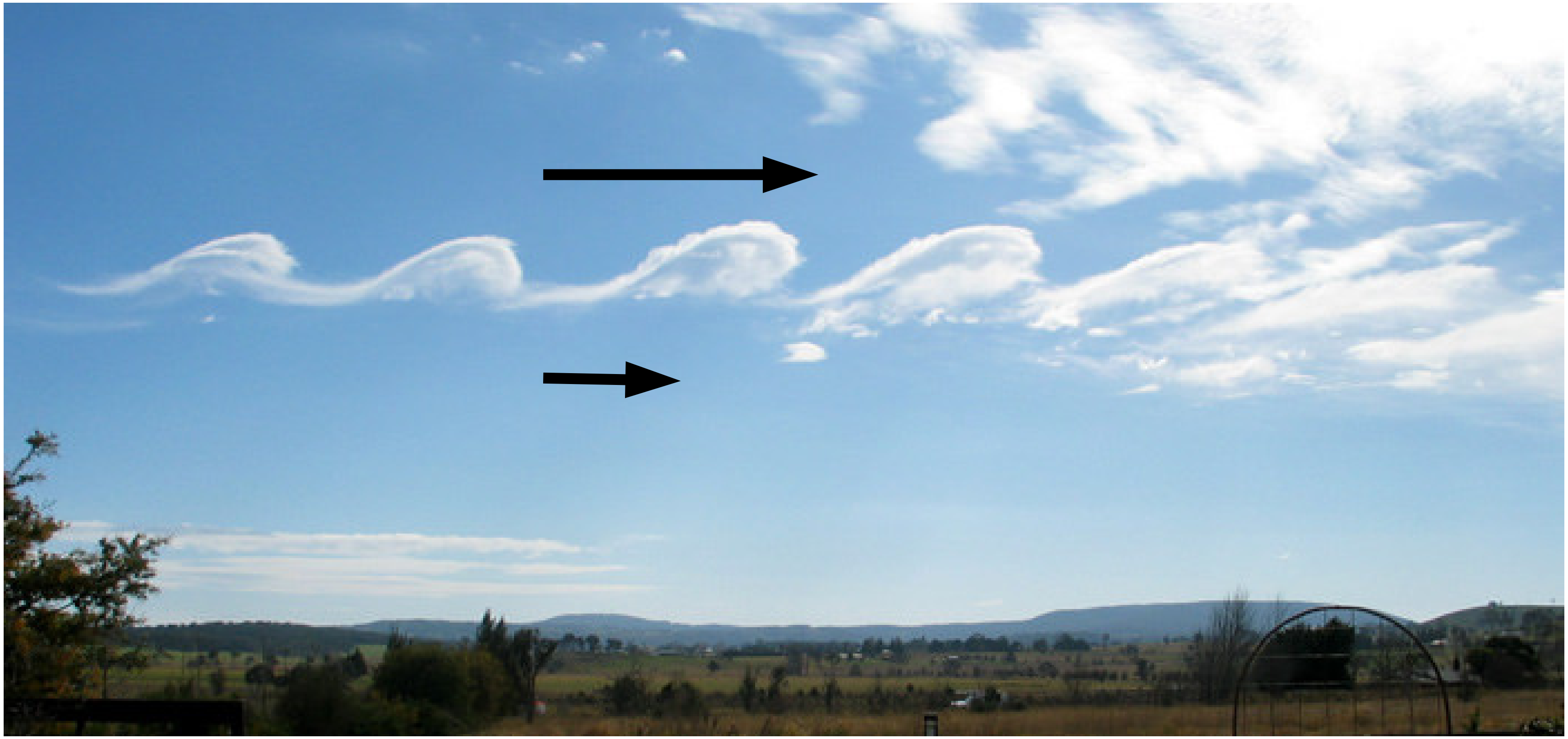




(Fabian et al. 2000)

- Bubbles rise buoyantly through density difference.
- Bubbles' age is several tens of millions of years.

Kelvin-Helmholtz Instability



(GRAHAMUK/Wikimedia Commons)

- Bubbles should get disrupted.
- What is the reason for their stability?

Numerical Experiments

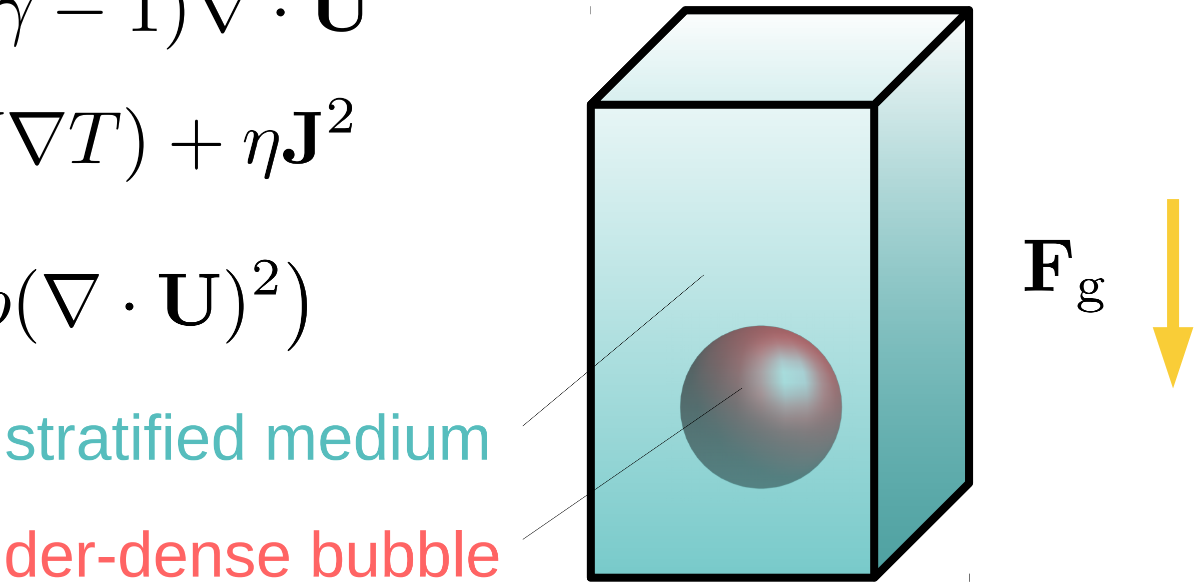
Full resistive magnetohydrodynamics simulations with the PencilCode.

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \eta \nabla^2 \mathbf{A}$$

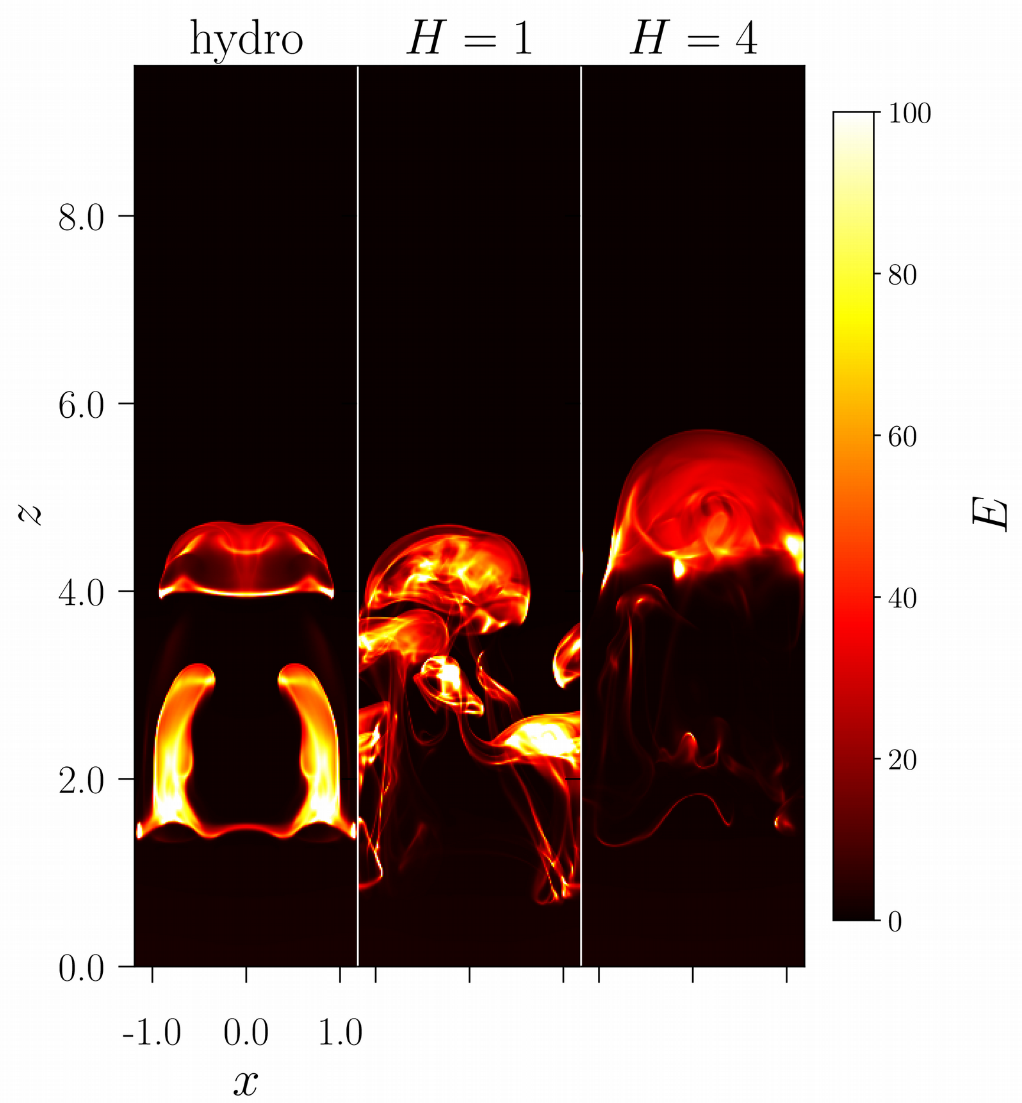
$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \nabla \left(\frac{\ln T}{\gamma} \ln \rho \right) + \mathbf{J} \times \mathbf{B}/\rho - \mathbf{g} + \mathbf{F}_{\mathrm{visc}}$$

$$\frac{\partial \ln T}{\partial t} = -\mathbf{U} \cdot \nabla \ln T - (\gamma - 1) \nabla \cdot \mathbf{U} + \frac{1}{\rho c_V T} \left(\nabla \cdot (K \nabla T) + \eta \mathbf{J}^2 + 2\rho\nu \mathsf{S} \otimes \mathsf{S} + \zeta\rho(\nabla \cdot \mathbf{U})^2 \right)$$

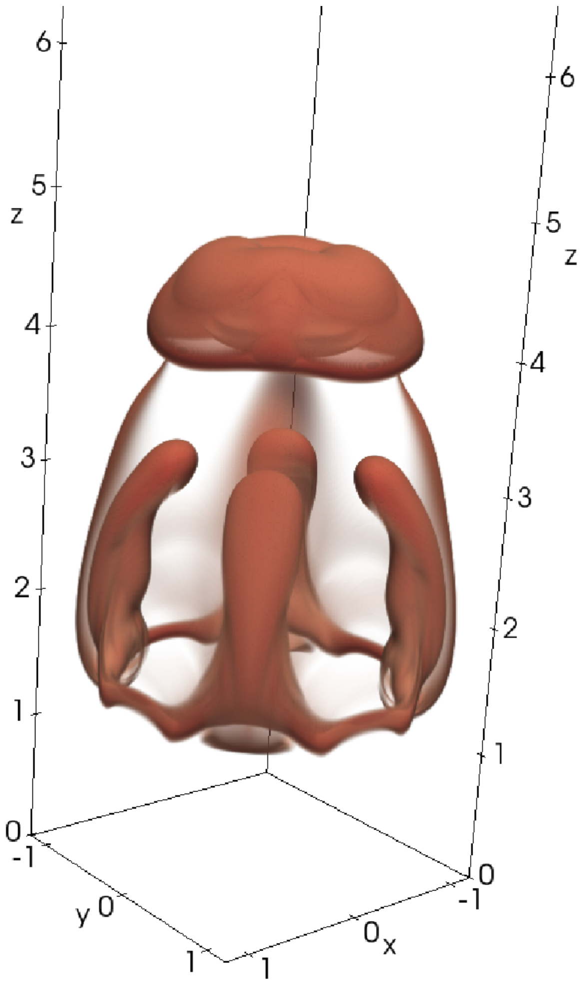
$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\nabla \cdot \mathbf{U}$$



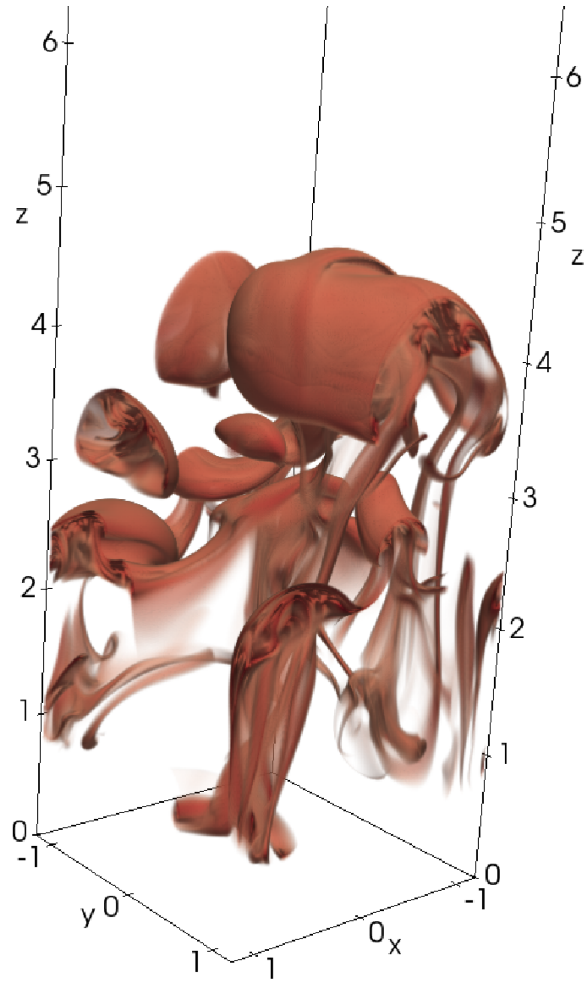
Thermal Emission



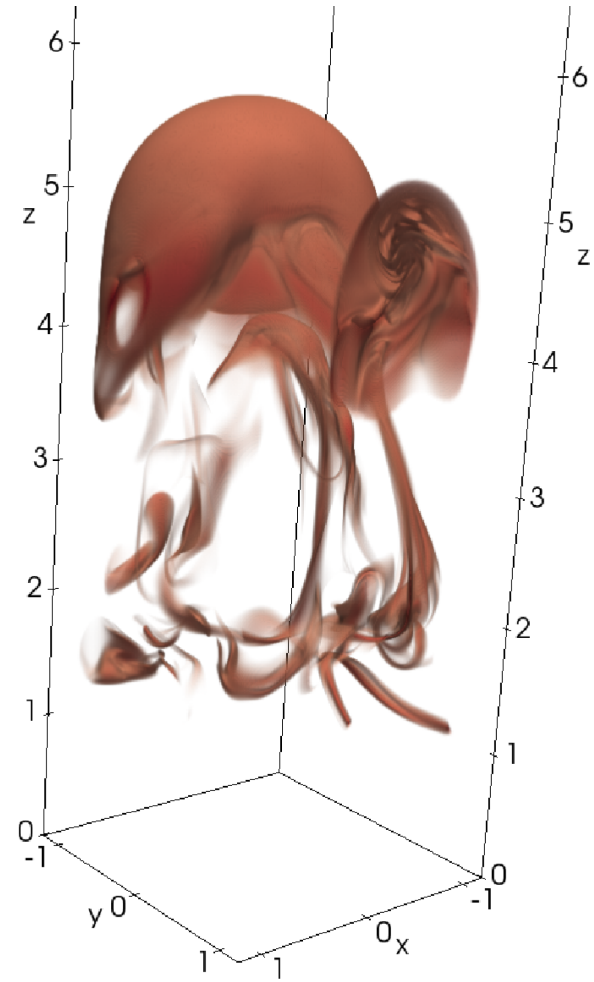
Temperature Iso-Surfaces



hydro

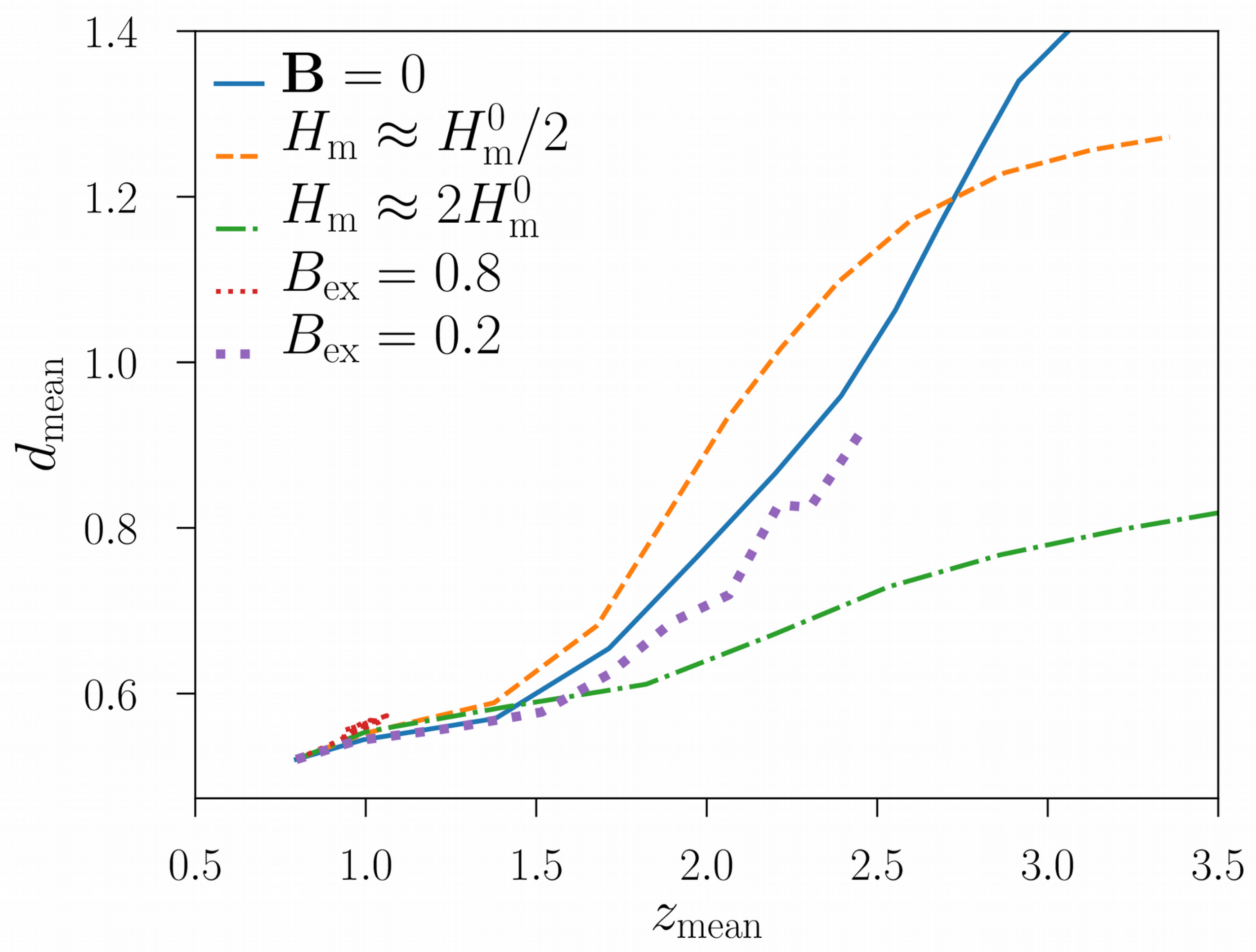


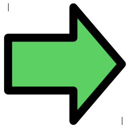
low helicity



high helicity

Bubble Coherence





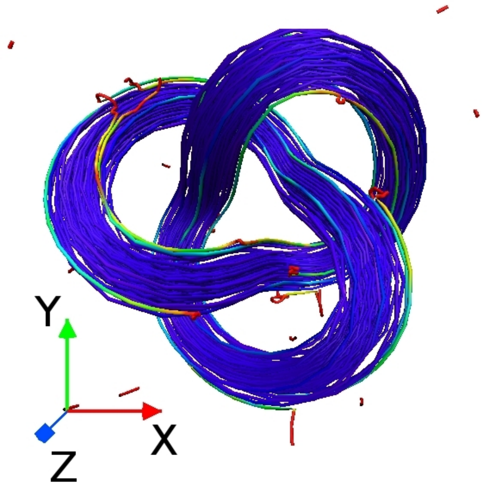
Helical magnetic fields can stabilize the bubbles.

Conclusions

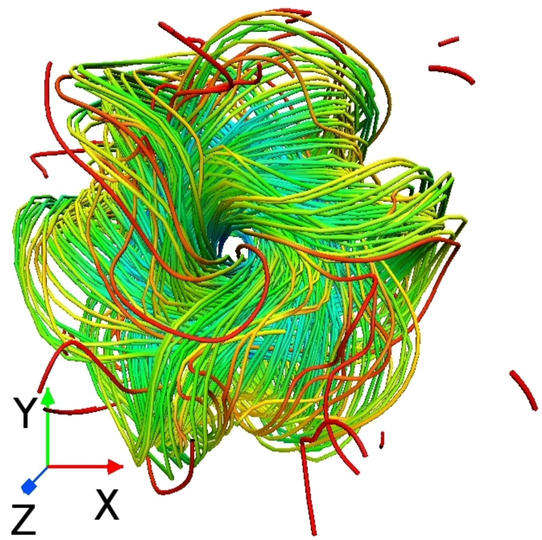
- Magnetic helicity as constraint on plasma dynamics.
- Further topological constraint: fixed point index, field line helicity, quadratic helicities.
- Topology preserving relaxation of magnetic fields.
- Current increases strongly with field complexity.
- Ideal relaxation of knots and braids.

simon.candelaresi@gmail.com

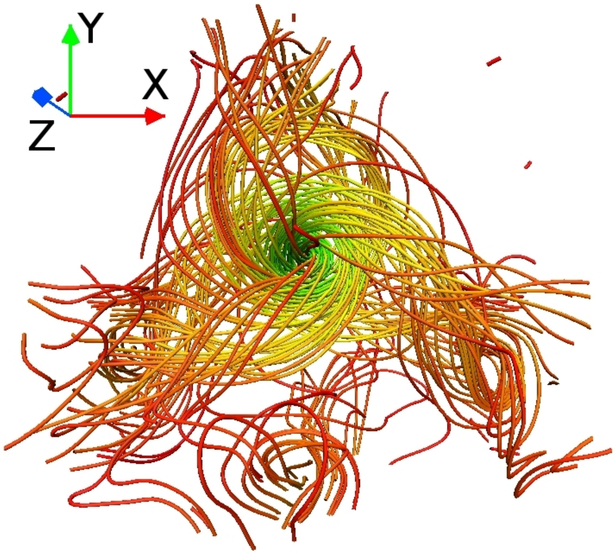
N-foil knots



$t = 0$



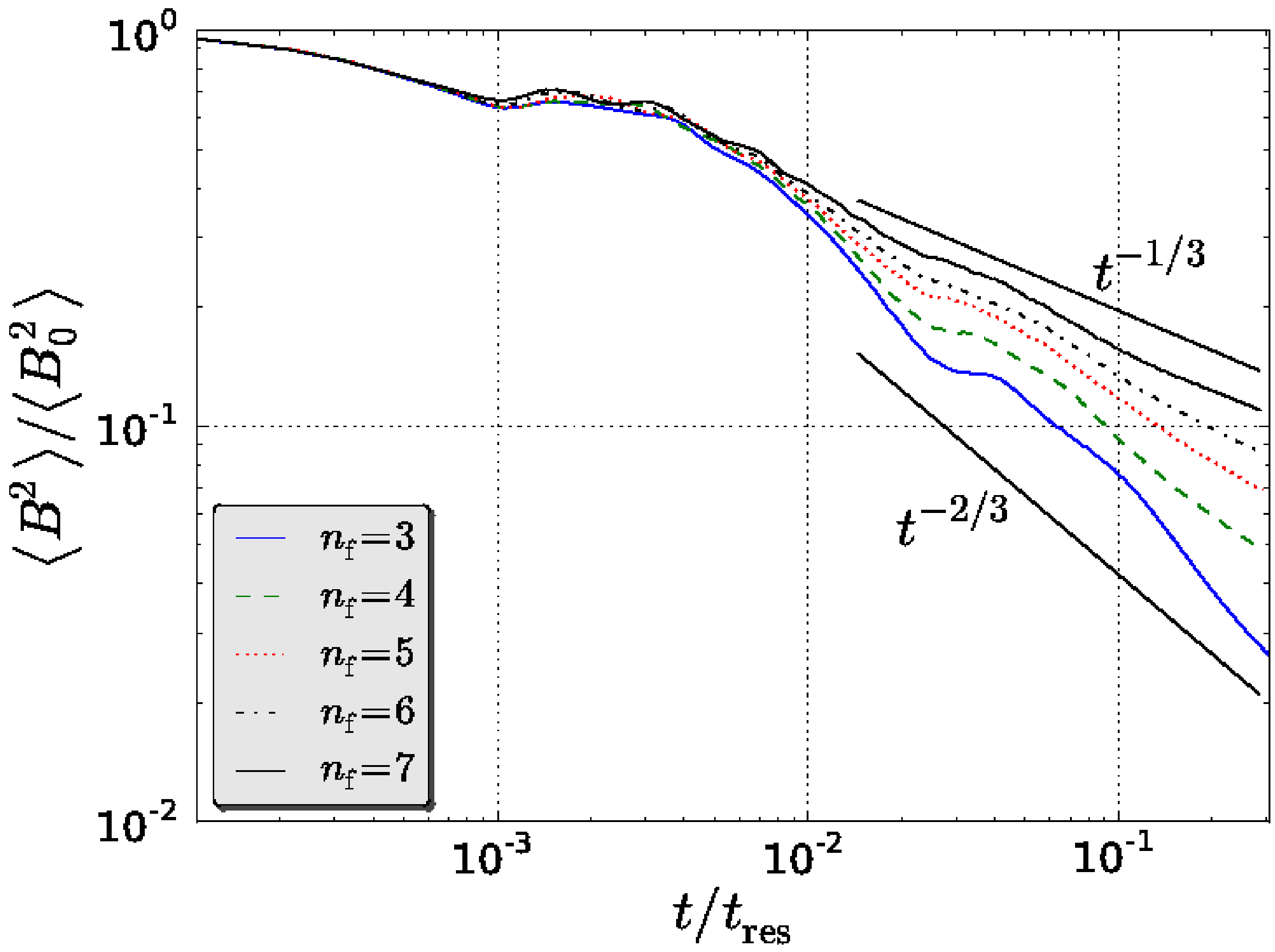
$t = 6$



$t = 39$

- Magnetic helicity is approximately conserved.
- Self-linking is transformed into twisting after reconnection.

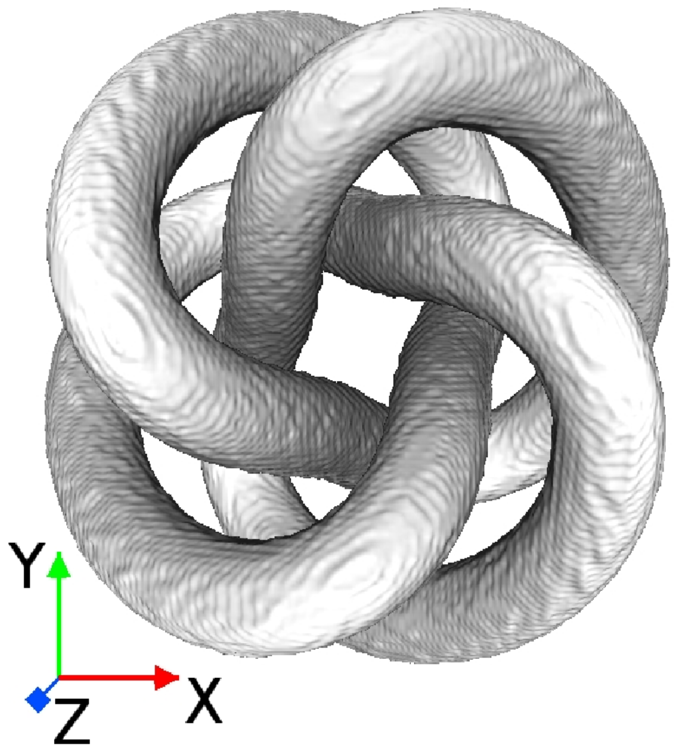
N-foil knots



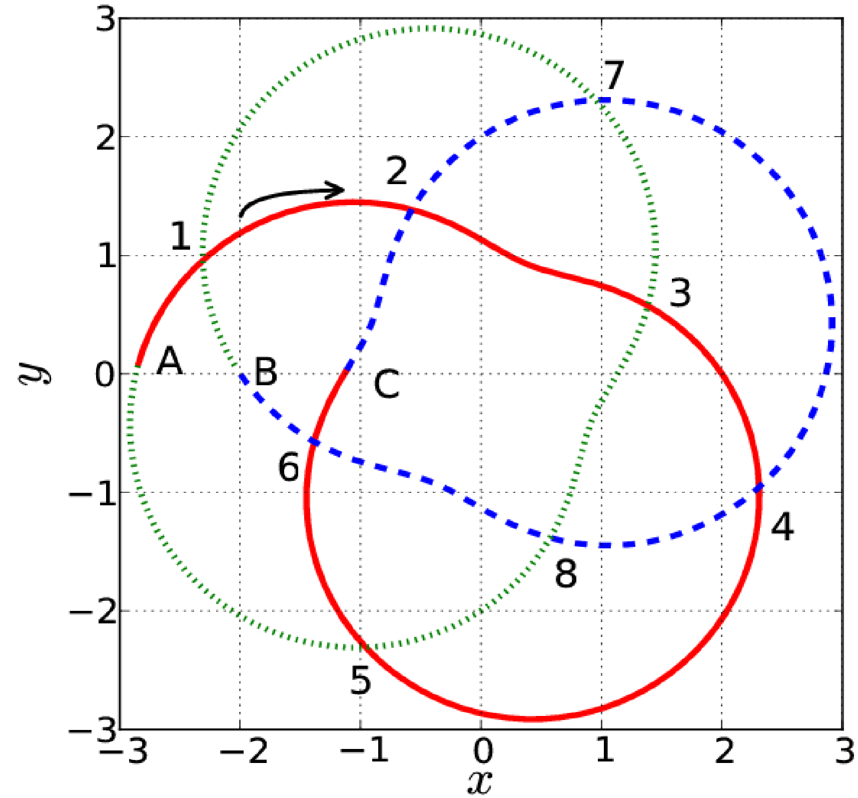
Slower decay for higher $n_{\rm f}$.

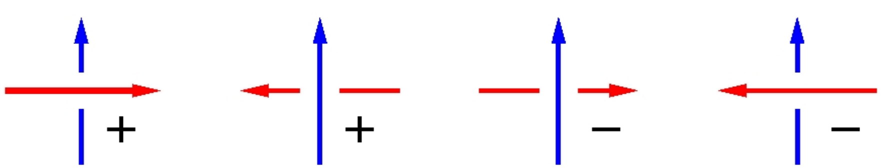
Braid Representation

need $B_z > 0$ → braid representation of knots and links

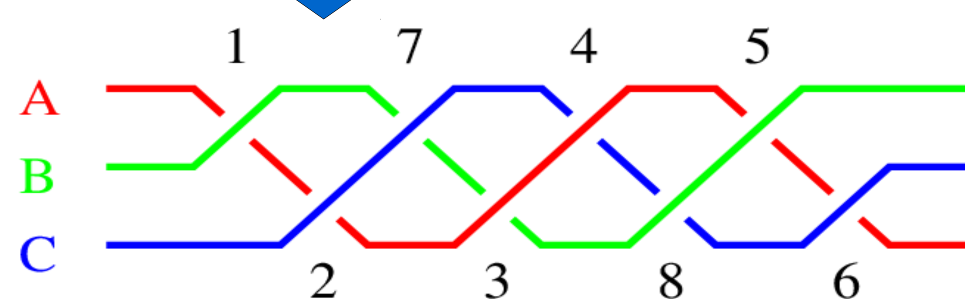


4-foil knot





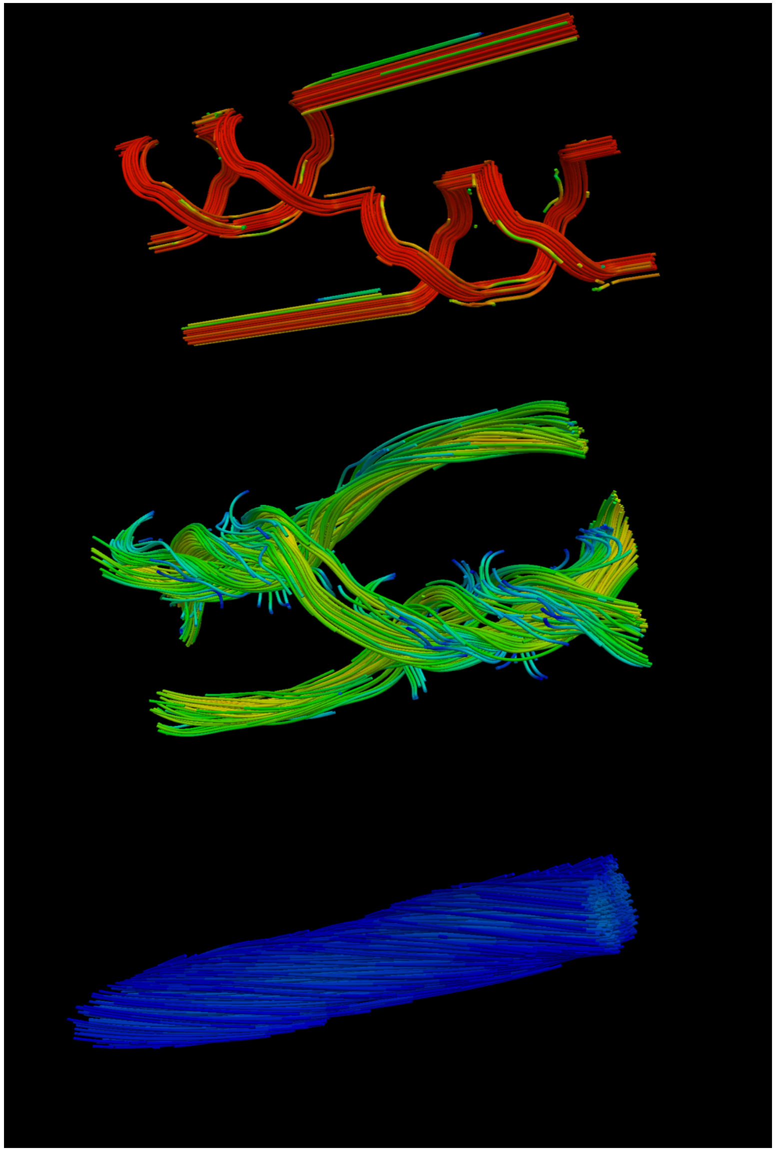
$$n_{\text{linking}} = (n_+ - n_-)/2$$



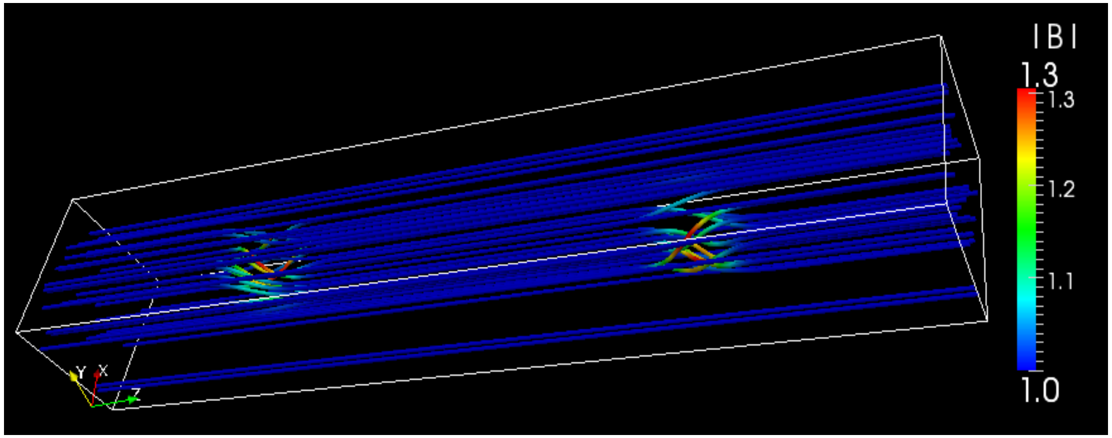
Magnetic Braid Configurations

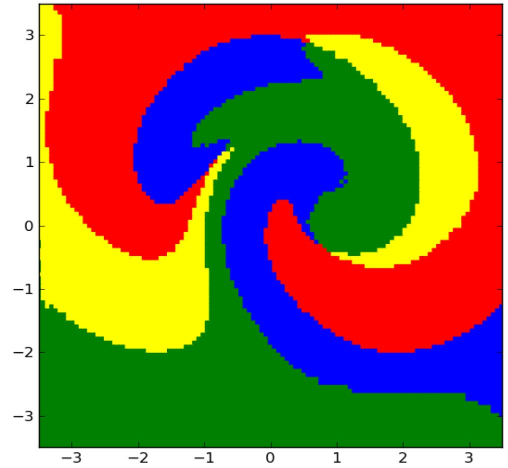
AAA (trefoil knot)

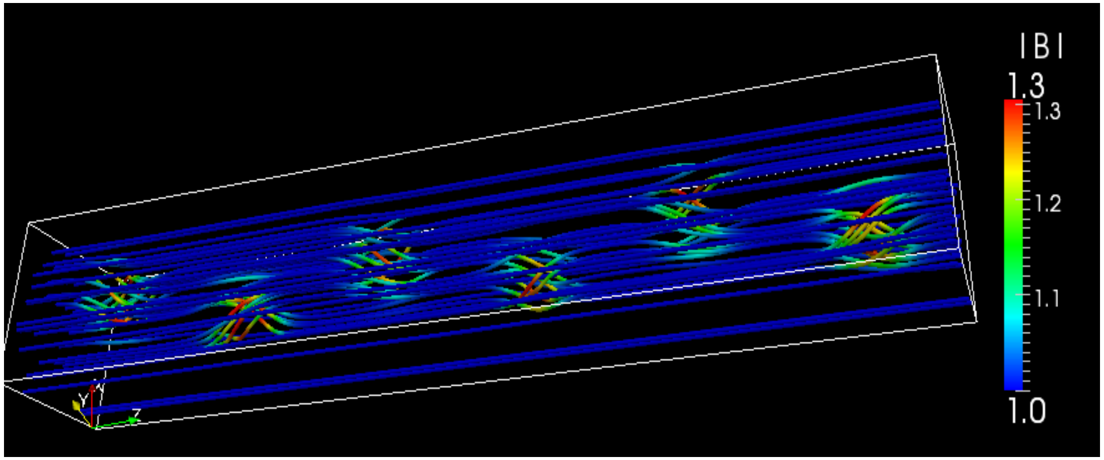
AABB (Borromean rings)

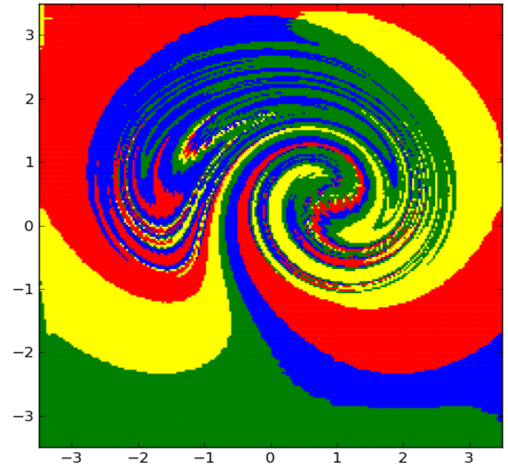


Field Line Tracing









Generalized flux function:

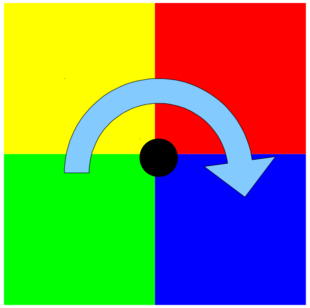
$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

Reconnection rate:

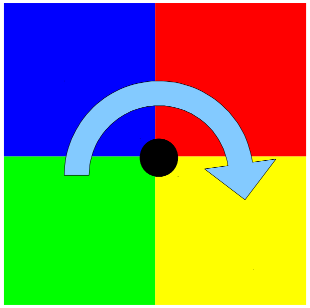
$$\sum_i \frac{\mathrm{d}\mathcal{A}(\mathbf{x}_i)}{\mathrm{d}t}$$

Fixed Point Index

Sign t_i of fixed point i :

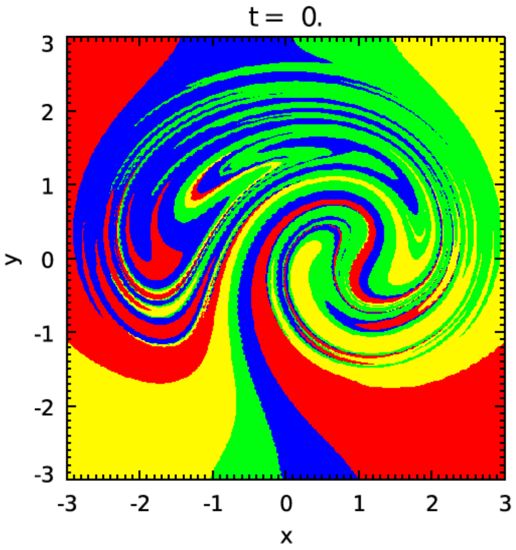


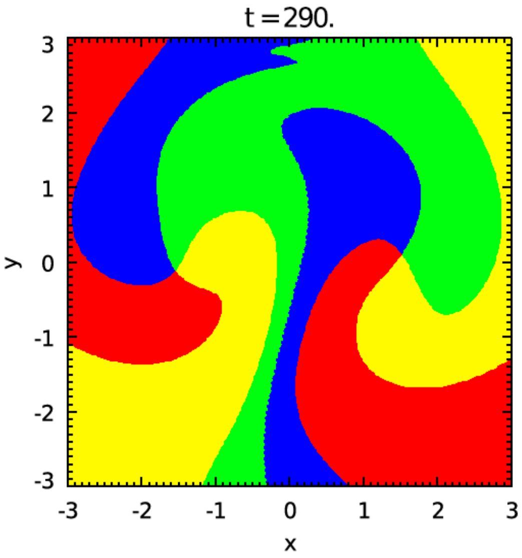
$t_i = +1$

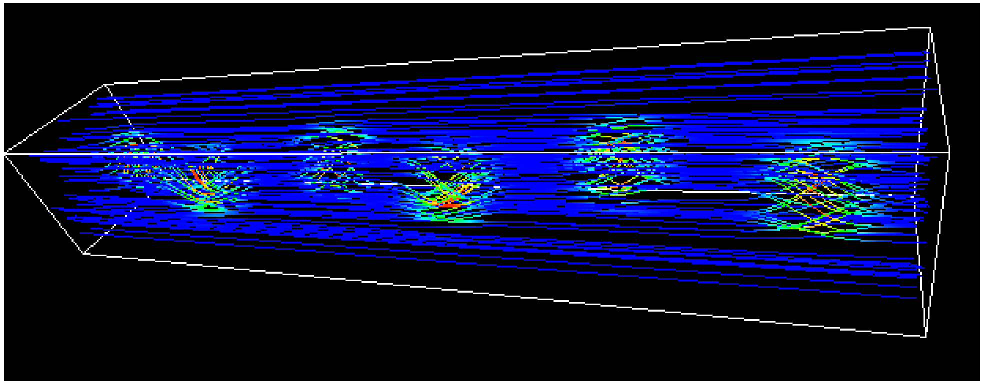


$t_i = -1$

Fixed point index: $T = \sum_i t_i$ conserved for $\lim \eta \to 0$



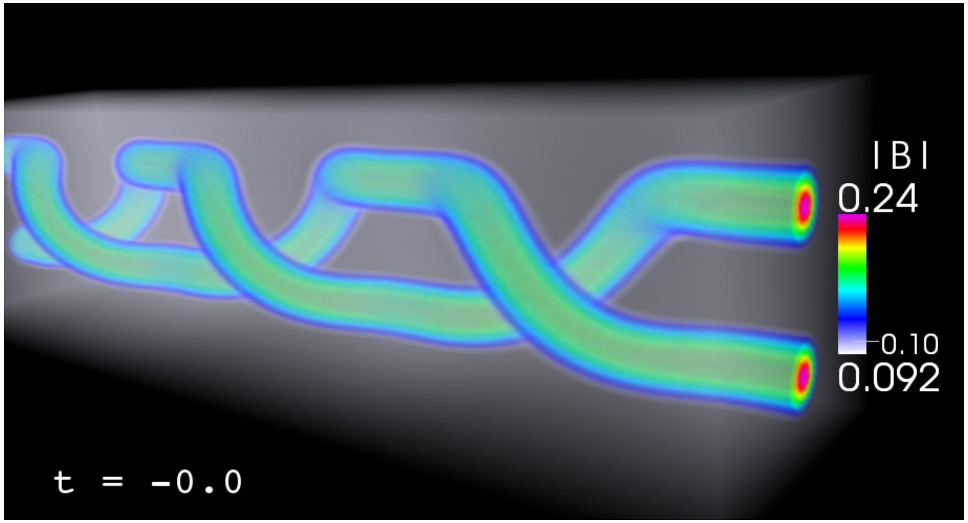


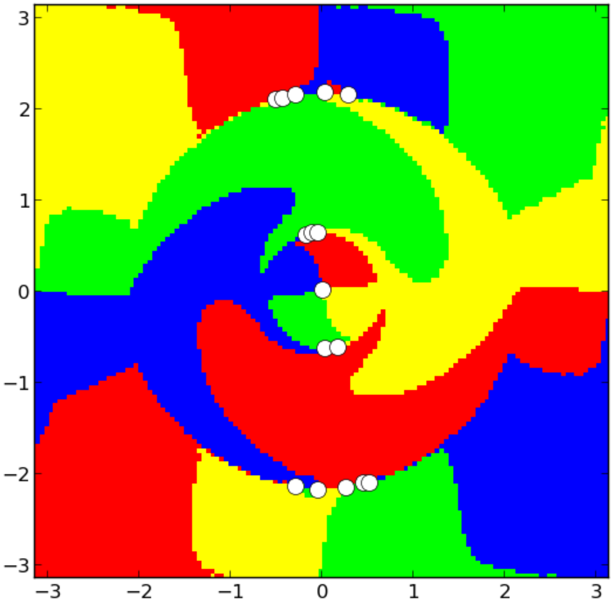


Taylor state is not reached
$\rightarrow$ T is additional constraint

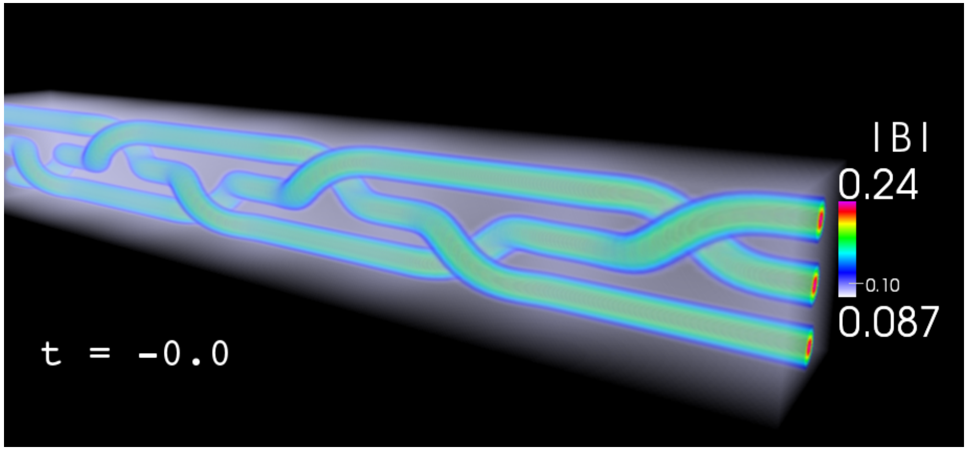
Knots as Braids

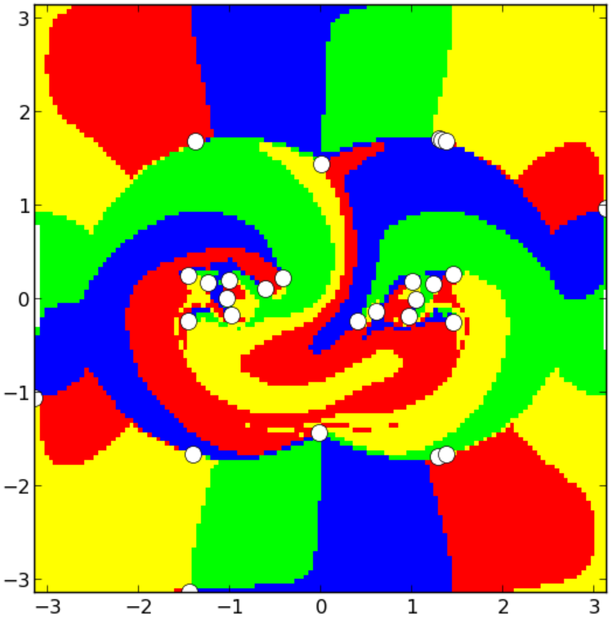
AAA, trefoil knot





AbAbAb, Borromean rings





Field's Environment

Magnetically dominated:

magnetic pressure >> thermal pressure

$$B^2/(2\mu_0) \gg nk_\mathrm{B}T$$

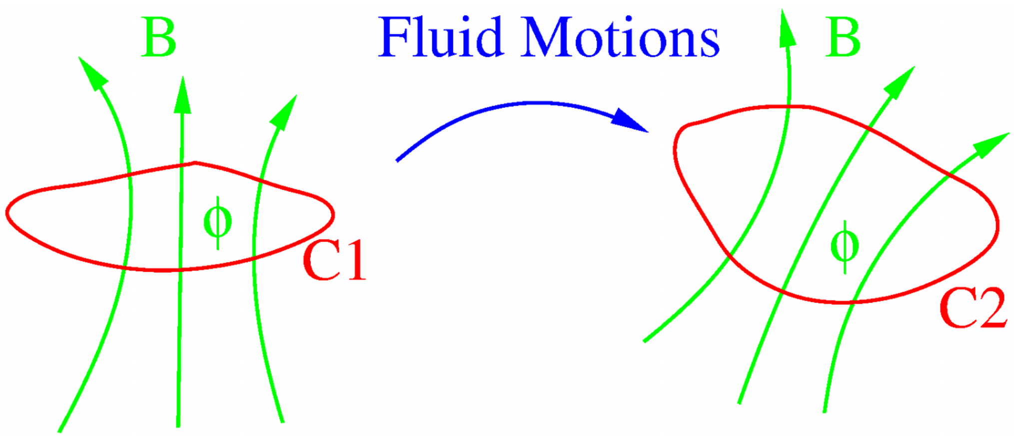
$$\beta = 2\mu_0 \frac{nk_\mathrm{B}T}{B^2} \ll 1$$

Solar corona: $\beta \approx 0.01$

Frozen-in magnetic flux:

magnetic resistivity small: $t_\mathrm{dissipation} \gg t_\mathrm{dynamical}$

⇨ Magnetic field is *frozen-in* to the fluid.



Batchelor (1950)

Ideal Field Relaxation

Ideal induction eq.: $$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

Frozen in magnetic field.

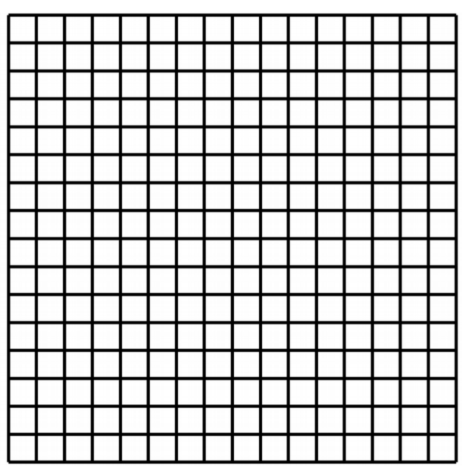
(Batchelor, 1950)

But: Numerical diffusion in finite difference Eulerian codes.

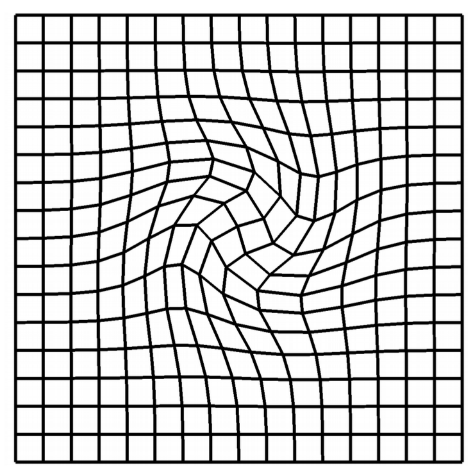
(Rembiasz, 2017)

Solution: Lagrangian description of moving fluid particles:

$$\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$$



$$\mathbf{x}(\mathbf{X}, t)$$



Ideal Field Relaxation

Field evolution: $$B_i(\mathbf{X},t) = \frac{1}{\Delta}\sum_{j=1}^{3}\frac{\partial x_i}{\partial X_j}B_j(\mathbf{X},0)$$

$$\Delta = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

Preserves topology and divergence-freeness.

Grid evolution: $$\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \mathbf{u}\left(\mathbf{x}(\mathbf{X},t),t\right)$$

Magneto-frictional term: $\mathbf{u} = \gamma \mathbf{J}\times\mathbf{B} \qquad \mathbf{J} = \nabla\times\mathbf{B}$

$$\Rightarrow \frac{\mathrm{d}E_{\mathrm{M}}}{\mathrm{d}t} < 0$$

(Craig and Sneyd 1986)

Numerical Curl Operator

Compute $\mathbf{J} = \nabla \times \mathbf{B}$ on a distorted grid:

$$\frac{\partial B_i}{\partial x_j} = X_{\alpha,j}(x_{i,\alpha\beta} B^0_\beta \Delta^{-1} + x_{i,\beta} B^0_{\beta,\alpha} \Delta^{-1} - x_{i,\beta} B^0_\beta \Delta^{-2} \Delta_{,\alpha})$$

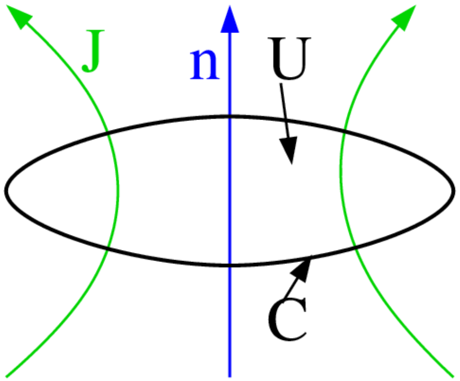
(Craig and Sneyd 1986)

$$B^0_i = B_i(0)$$

- Multiplication of several terms leads to high numerical errors.
- Current not divergence free: $\nabla \cdot \mathbf{J} \neq 0$
- Only reaching a certain force-freeness. *(Pontin et al. 2009)*

Mimetic Numerical Operators

$$I = \int_U \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}S = \oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r}$$



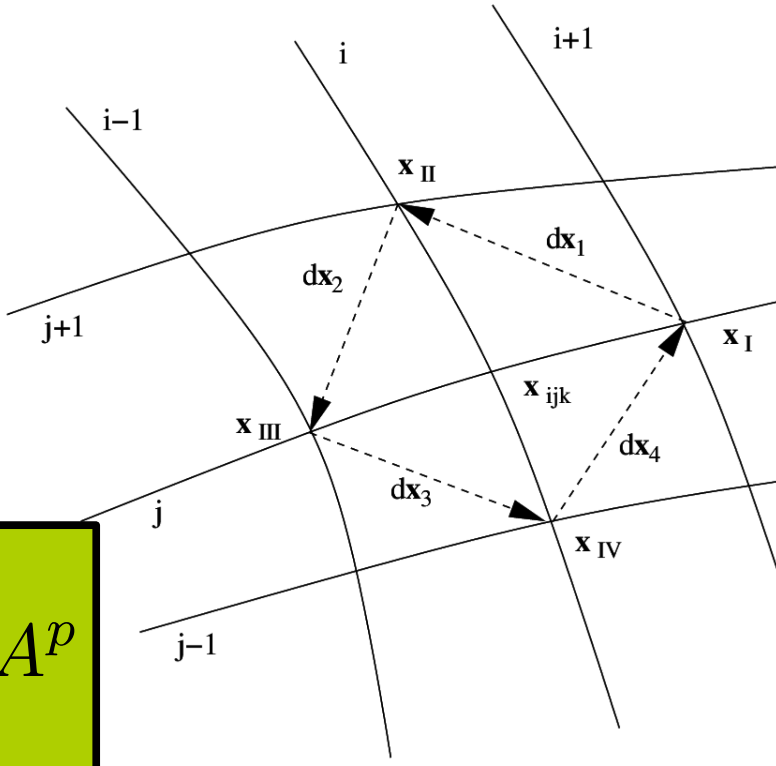
Discretized:

$$I \approx \mathbf{J}(\mathbf{X}_{ijk}) \cdot \mathbf{n} A = \sum_{r=1}^{4} \mathbf{B}_r \cdot \mathrm{d}\mathbf{x}_r$$

$$\mathbf{J}(\mathbf{X}_U) \approx \mathbf{J}(\mathbf{X_{ijk}}), \quad \mathbf{X}_U \in U$$

3 planes will give 3 l.i. normal vectors:

$$I^p = \mathbf{J}(\mathbf{X}_{ijk}) \cdot \mathbf{n}^p = \sum_{r=1}^{4} \mathbf{B}_r^p \cdot \mathrm{d}\mathbf{x}_r / A^p$$



Inversion yields $\mathbf{J}$ with $\nabla \cdot \mathbf{J} = 0$.

(Hyman, Shashkov 1997)