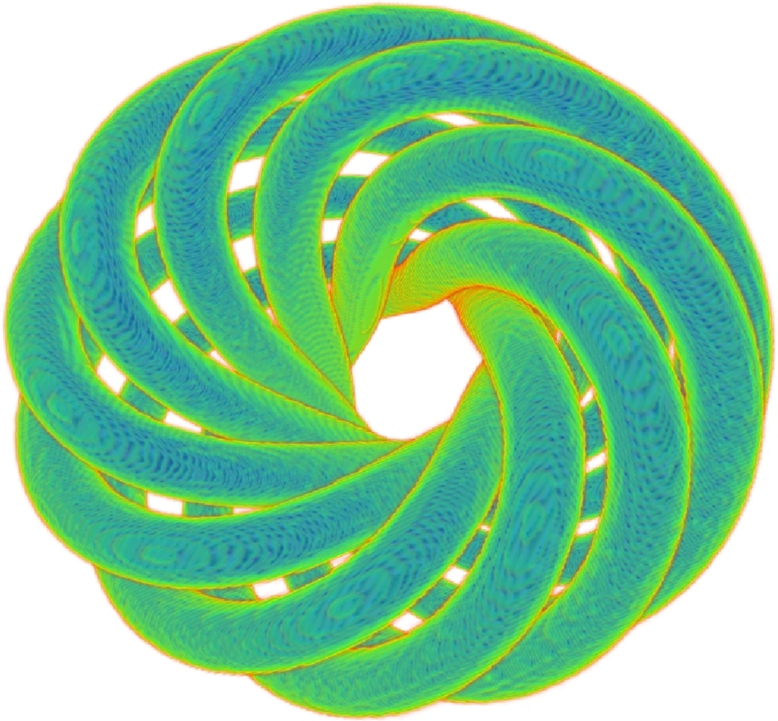
Topological constraints in magnetic field relaxation

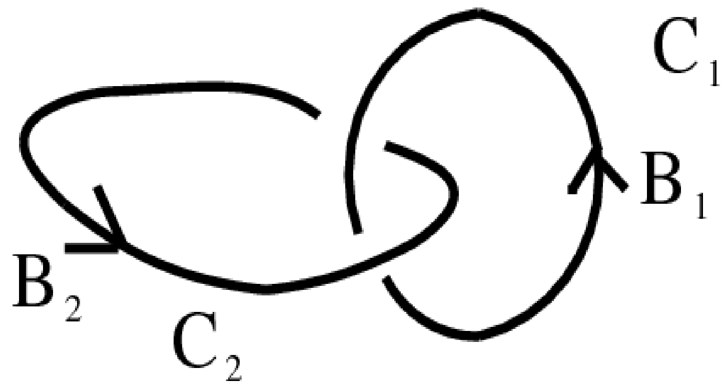
Simon Candelaresi



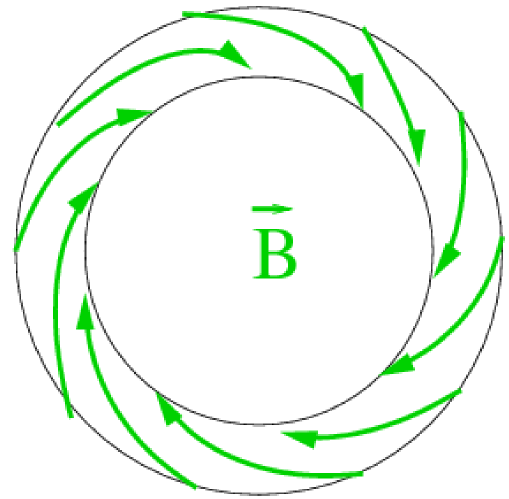
Outline

- Observations of topologically non-trivial magnetic fields (twist).
- Measure of topology.
- Magnetic helicity conservation, realizability condition.
- Equilibrium states: Woltjer and Taylor
- actual linking vs. magnetic helicity
- Fixed point index.
- Measures for the magnetic reconnection rate.

Topologies of Magnetic Fields

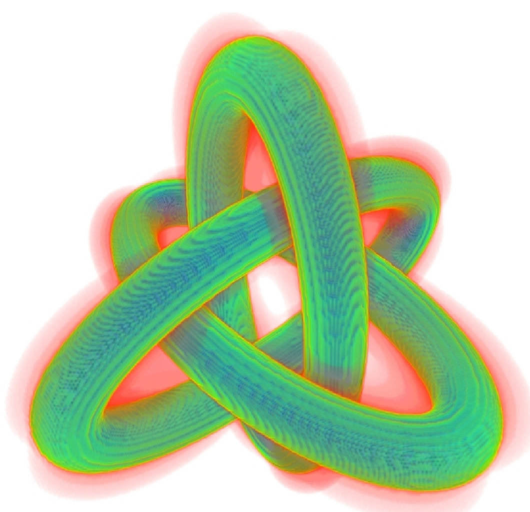


Hopf link

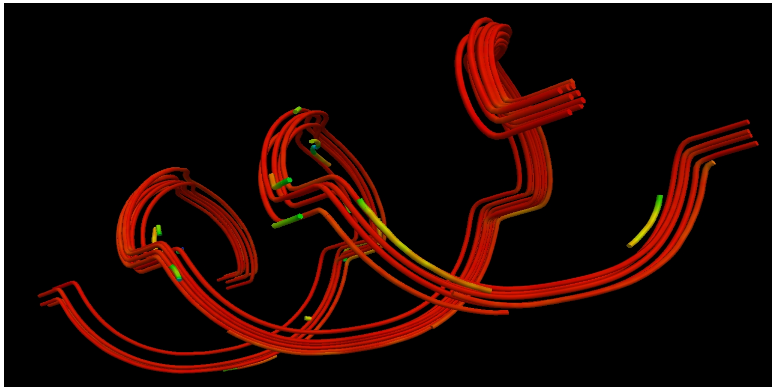


twisted field

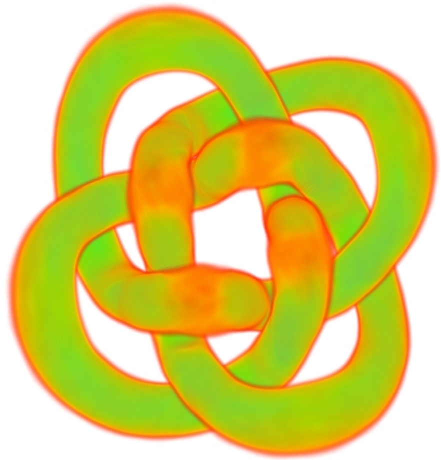
trefoil knot



Borromean rings

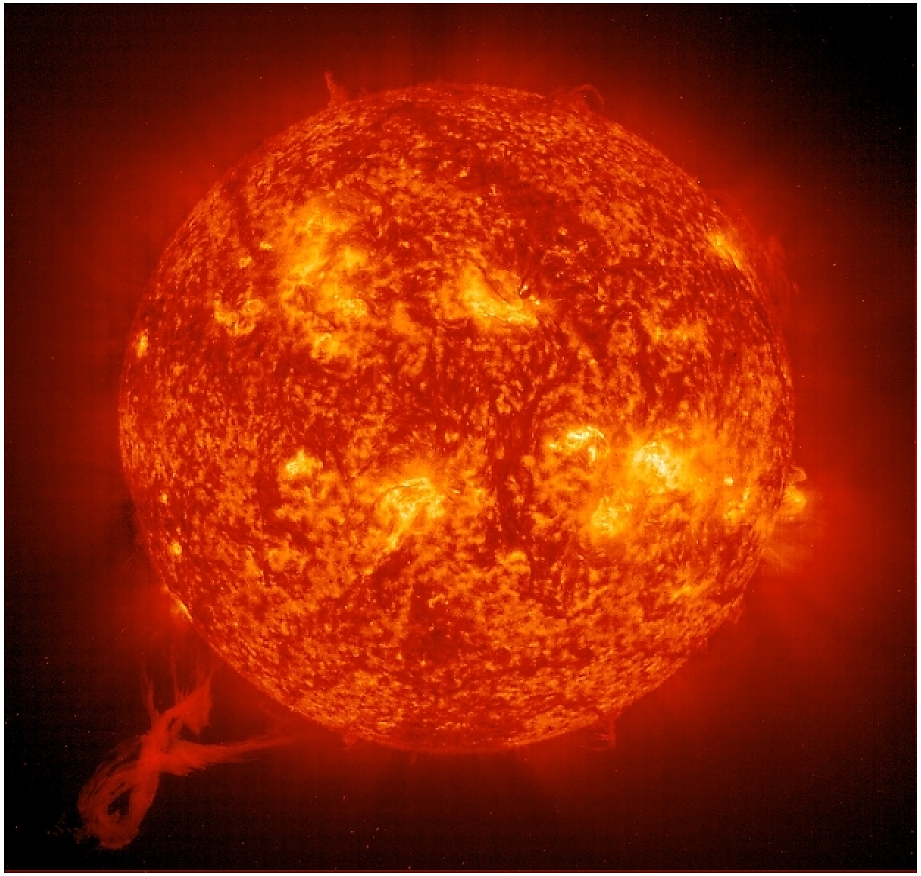


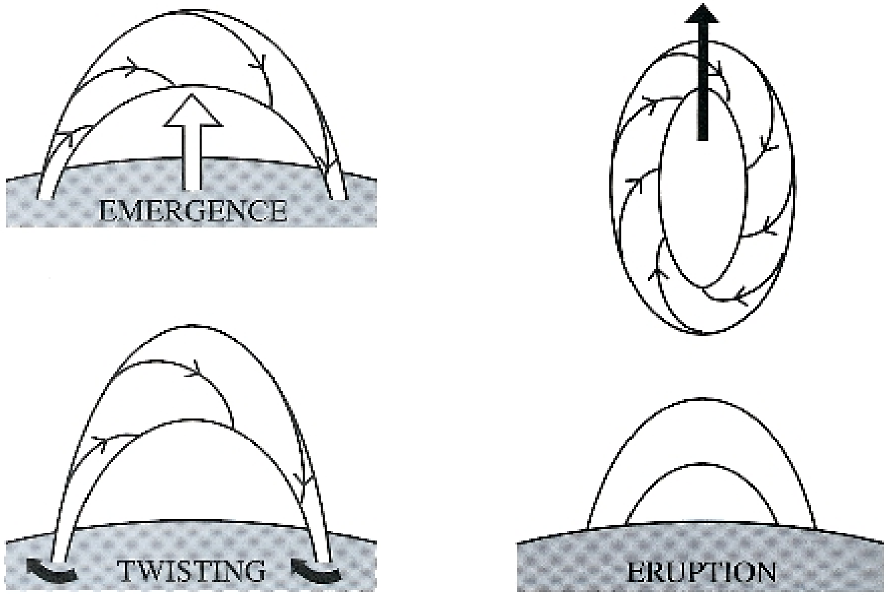
magnetic braid



IUCAA knot

Twisted Magnetic Fields





Twisted fields are more likely to erupt *(Canfield et al. 1999)*.

Twisted Field in the Sun

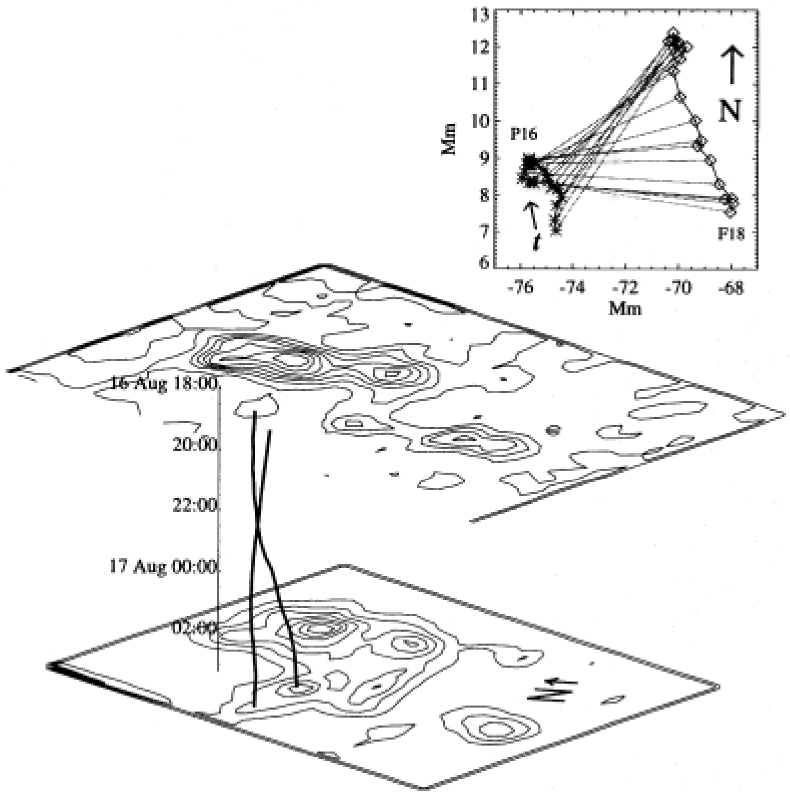
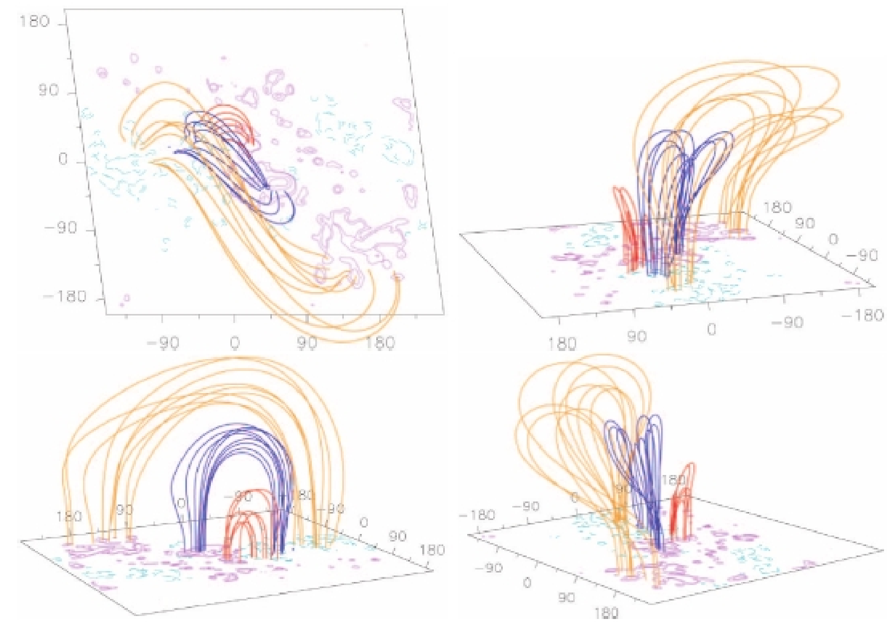


FIG. 3b

Magnetic bipoles' movement on the Sun's surface.
(Leka et al. 1996)



Force-free extrapolation of the photospheric magnetic field from 1999, August 21.
(Gibson et al. 2002)

Force free condition:

$$\nabla \times B = \alpha B$$

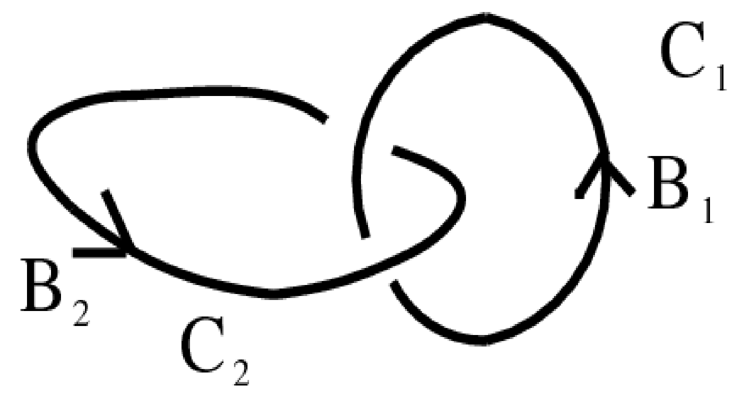
$$J \times B = 0$$

Magnetic Helicity

Measure for the topology:

$$H_{\mathrm{M}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 2n\phi_1\phi_2$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B} \qquad \phi_i = \int_{S_i} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$



$n =$ number of mutual linking

Conservation of magnetic helicity:

$$\lim_{\eta \to 0} \frac{\partial}{\partial t} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = 0 \qquad \eta = \text{magnetic resistivity}$$

Realizability condition:

$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

→ Magnetic energy is bound from below by magnetic helicity.

Equilibrium States

Ideal MHD: $\eta = 0$

Induction equation: $\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B})$

Task: Find the state with minimal energy.
Constraint: magnetic helicity conservation

	constraint	equilibrium
Woltjer (1958):	$\frac{\partial}{\partial t} \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha \boldsymbol{B}$
Taylor (1974):	$\frac{\partial}{\partial t} \int_{\tilde{V}} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha(a, b) \boldsymbol{B}$

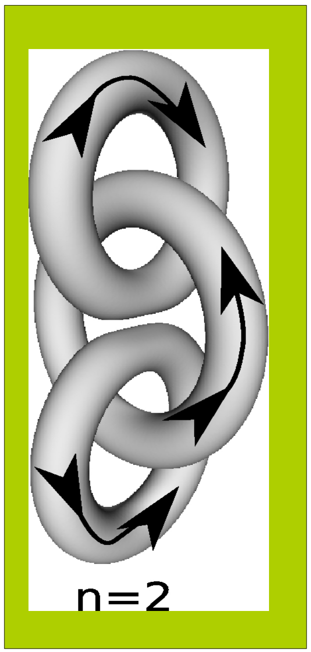
$\alpha(a, b)$: constant along field line

V total volume $\quad \tilde{V}$ volume along magnetic field line

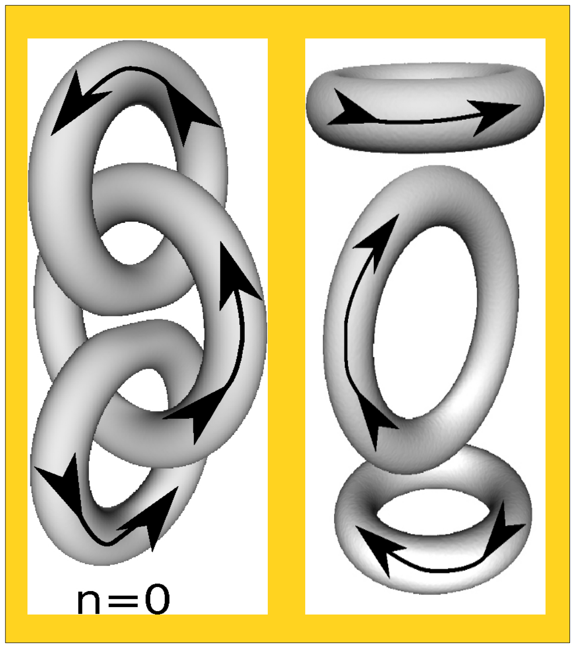
Interlocked Flux Rings

actual linking vs. magnetic helicity

$H_{\mathrm{M}} \neq 0$



$H_{\mathrm{M}} = 0$



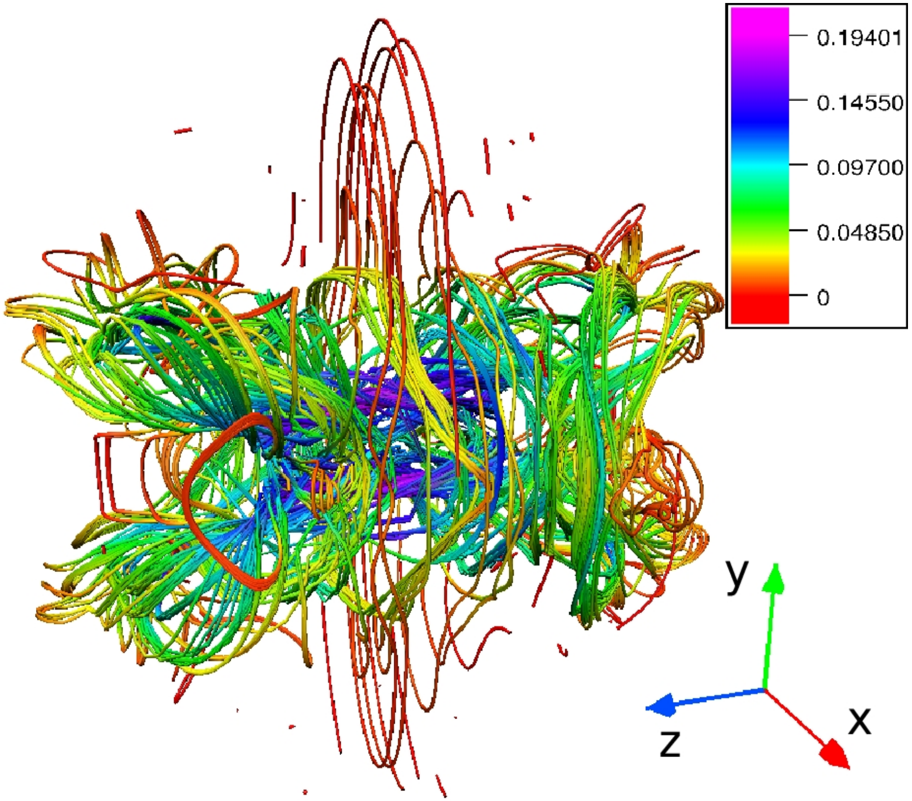
- initial condition: flux tubes
- isothermal compressible gas
- viscous medium
- periodic boundaries

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{U} \times \boldsymbol{B} + \eta \nabla^2 \boldsymbol{A} \qquad \frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \boldsymbol{U}$$

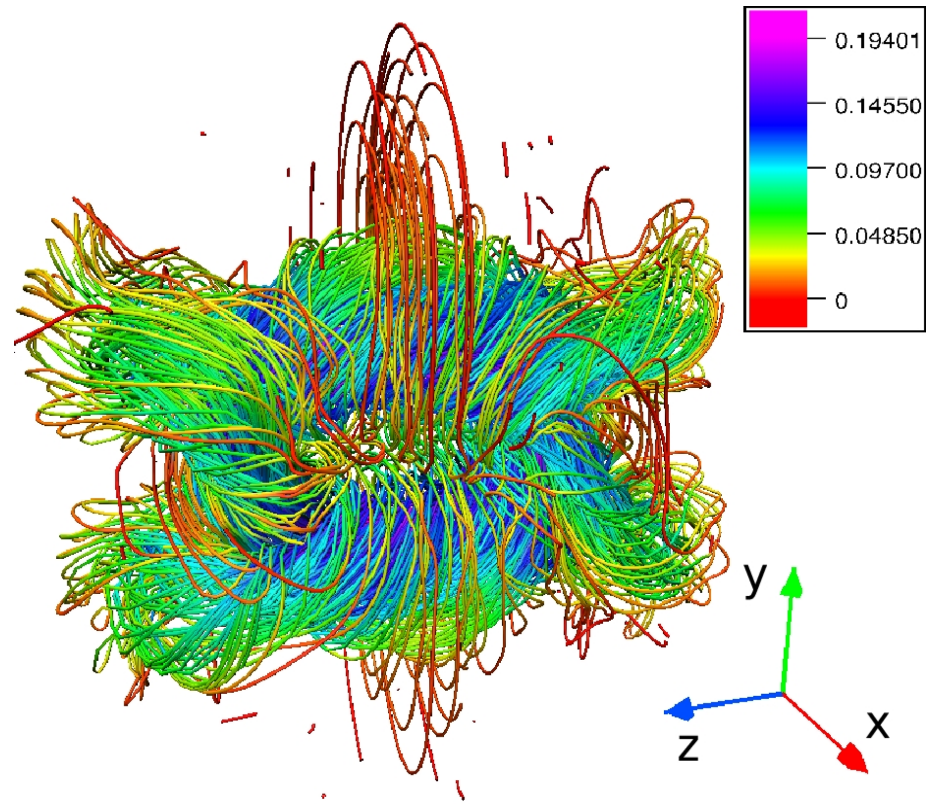
$$\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \boldsymbol{\nabla} \ln \rho + \boldsymbol{J} \times \boldsymbol{B}/\rho + \boldsymbol{F}_{\mathrm{visc}}$$

Interlocked Flux Rings

$\tau = 4$

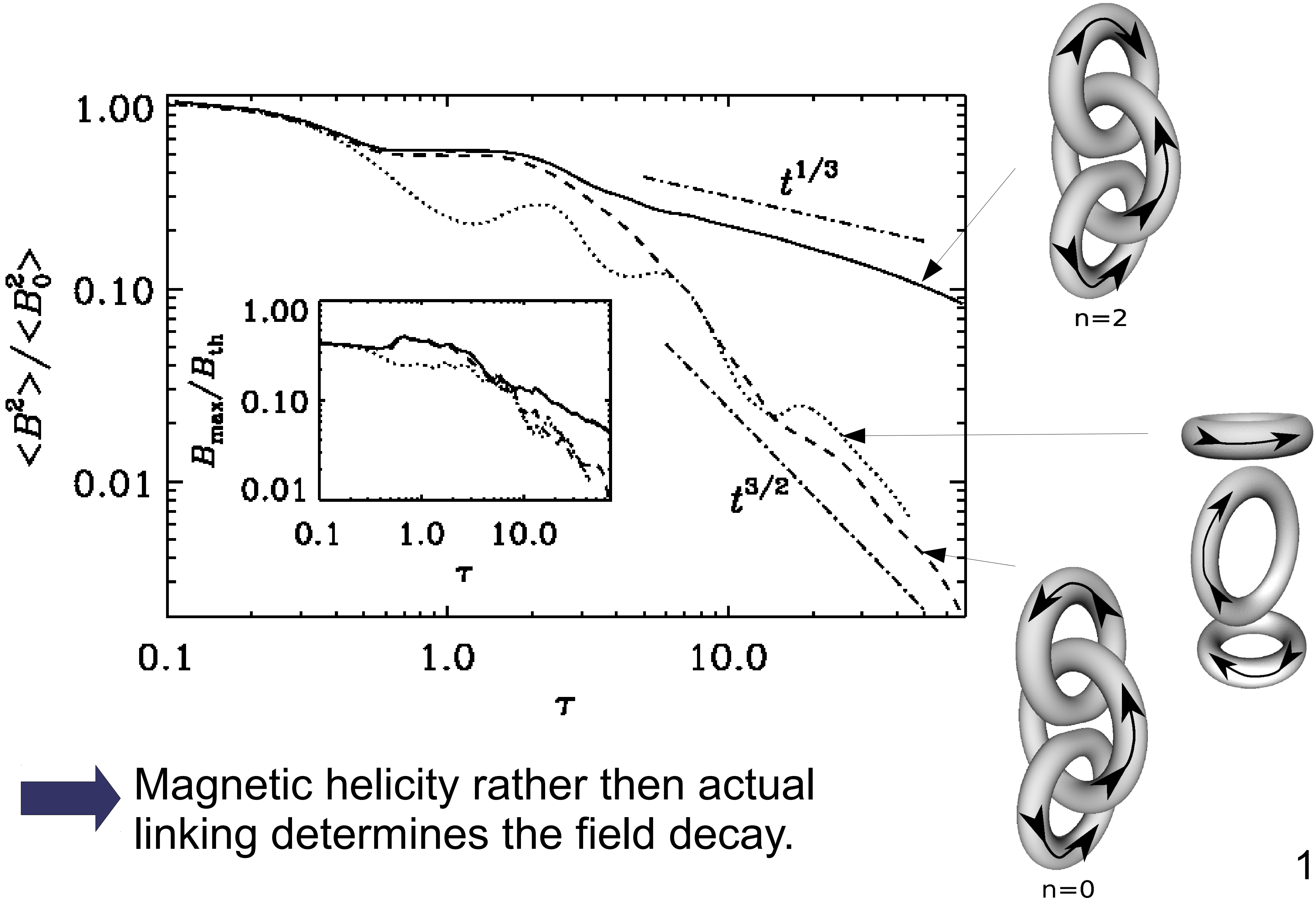


$H_{\mathrm{M}} = 0$



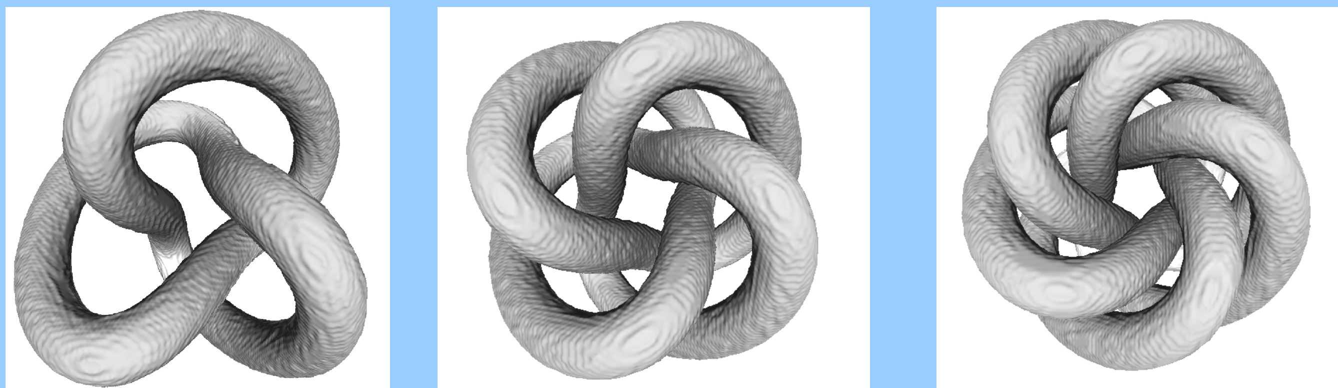
$H_{\mathrm{M}} \neq 0$

Interlocked Flux Rings

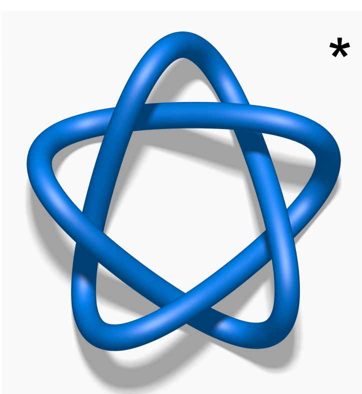


Magnetic helicity rather then actual linking determines the field decay.

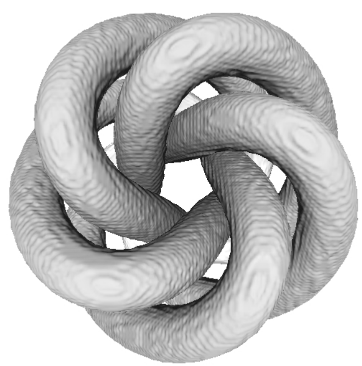
N-foil Knots



3-foil 4-foil 5-foil 6-foil 7-foil



$\neq$

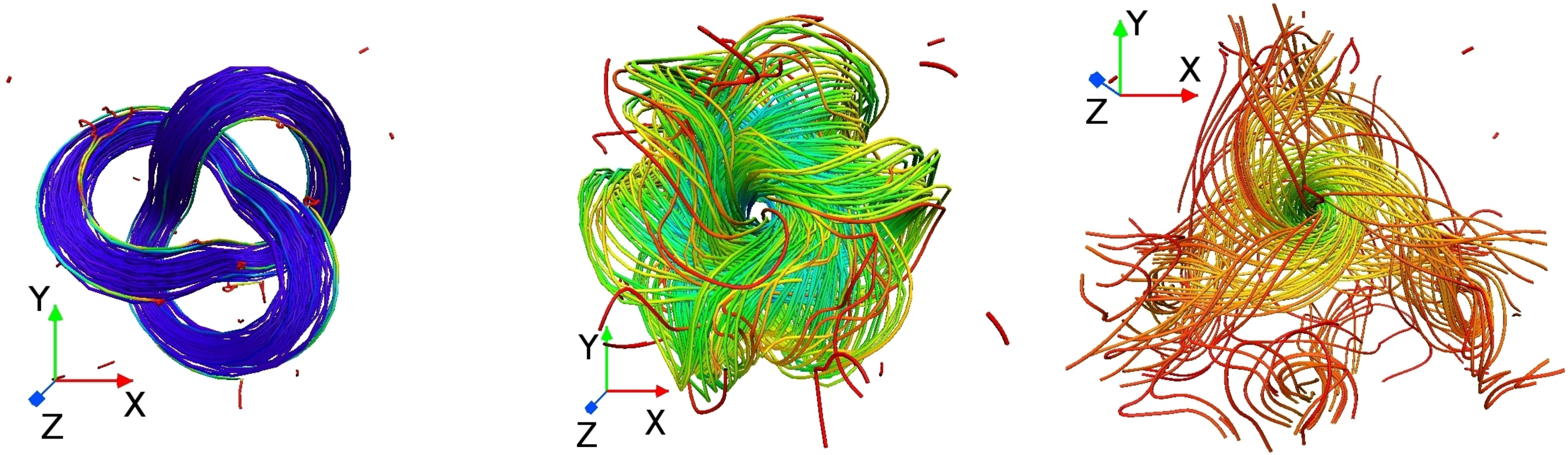


$$x(s) = \begin{pmatrix} (C + \sin s n_{\mathrm{f}}) \sin[s(n_{\mathrm{f}} - 1)] \\ (C + \sin s n_{\mathrm{f}}) \cos[s(n_{\mathrm{f}} - 1)] \\ D \cos s n_{\mathrm{f}} \end{pmatrix}$$

cinquefoil knot

* from Wikipedia, author: Jim.belk

N-foil Knots

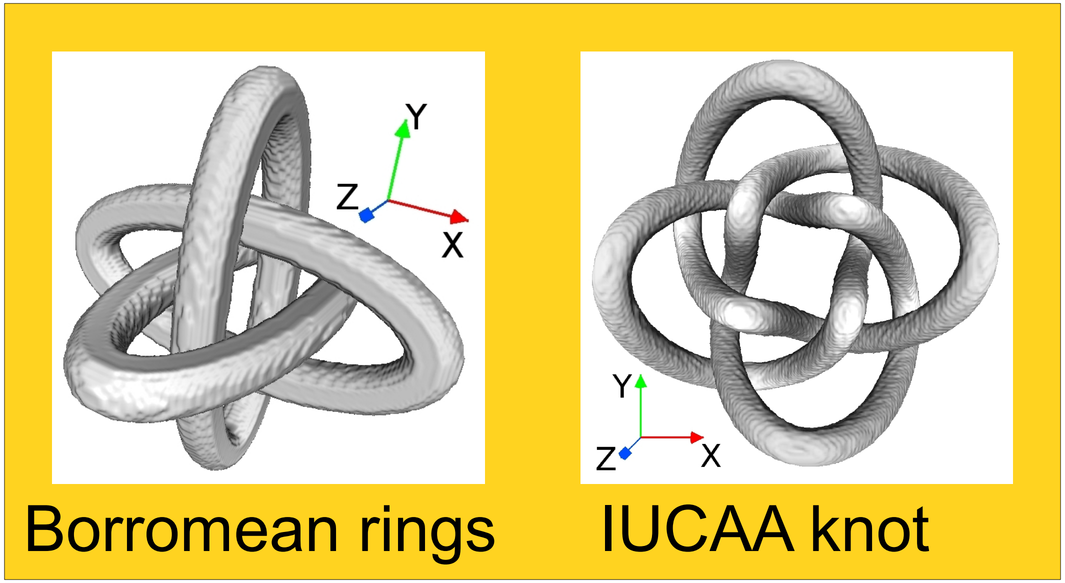


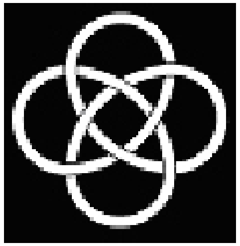
$t = 0$ $\quad$ $t = 6$ $\quad$ $t = 39$

- Magnetic helicity is approximately conserved.
- Self-linking is transformed into twisting after reconnection.

IUCAA Knot and Borromean Rings

- Is magnetic helicity sufficient?
- Higher order invariants?

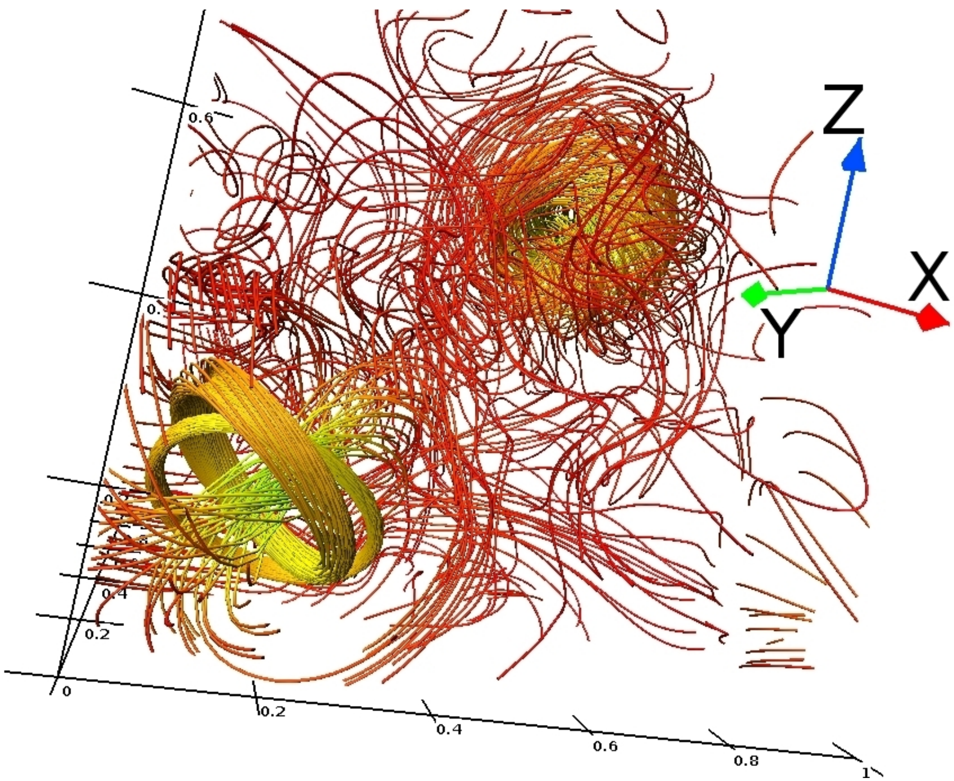




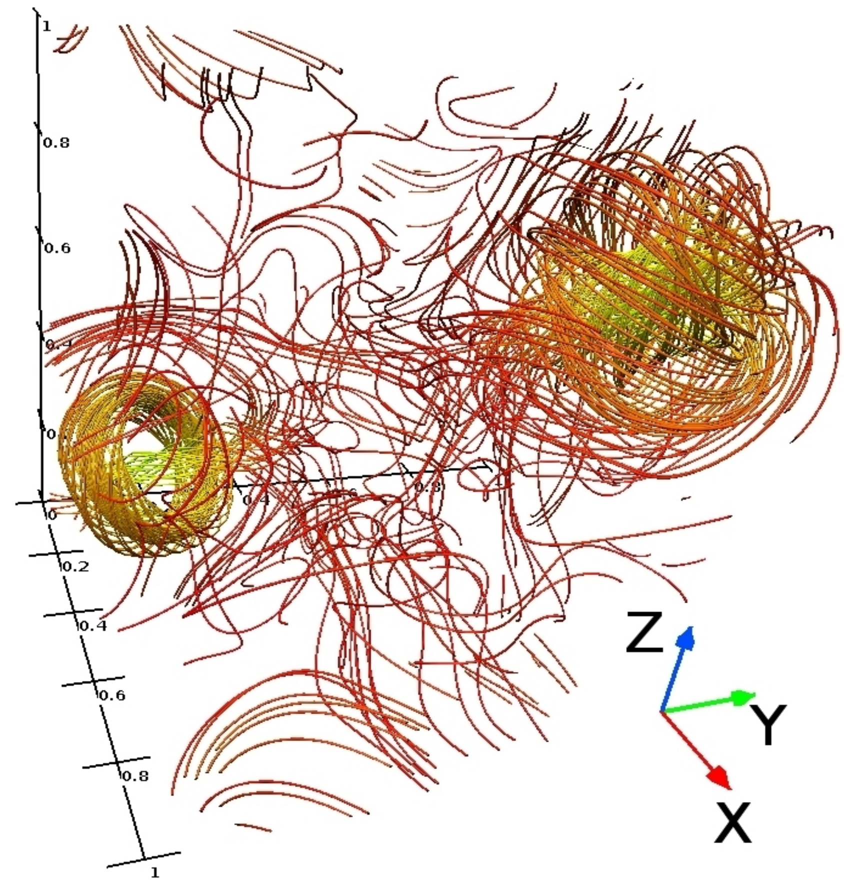
$$H_{\mathrm{M}} = 0$$

IUCAA = The Inter-University Centre for Astronomy and Astrophysics, Pune, India

Reconnection Characteristics



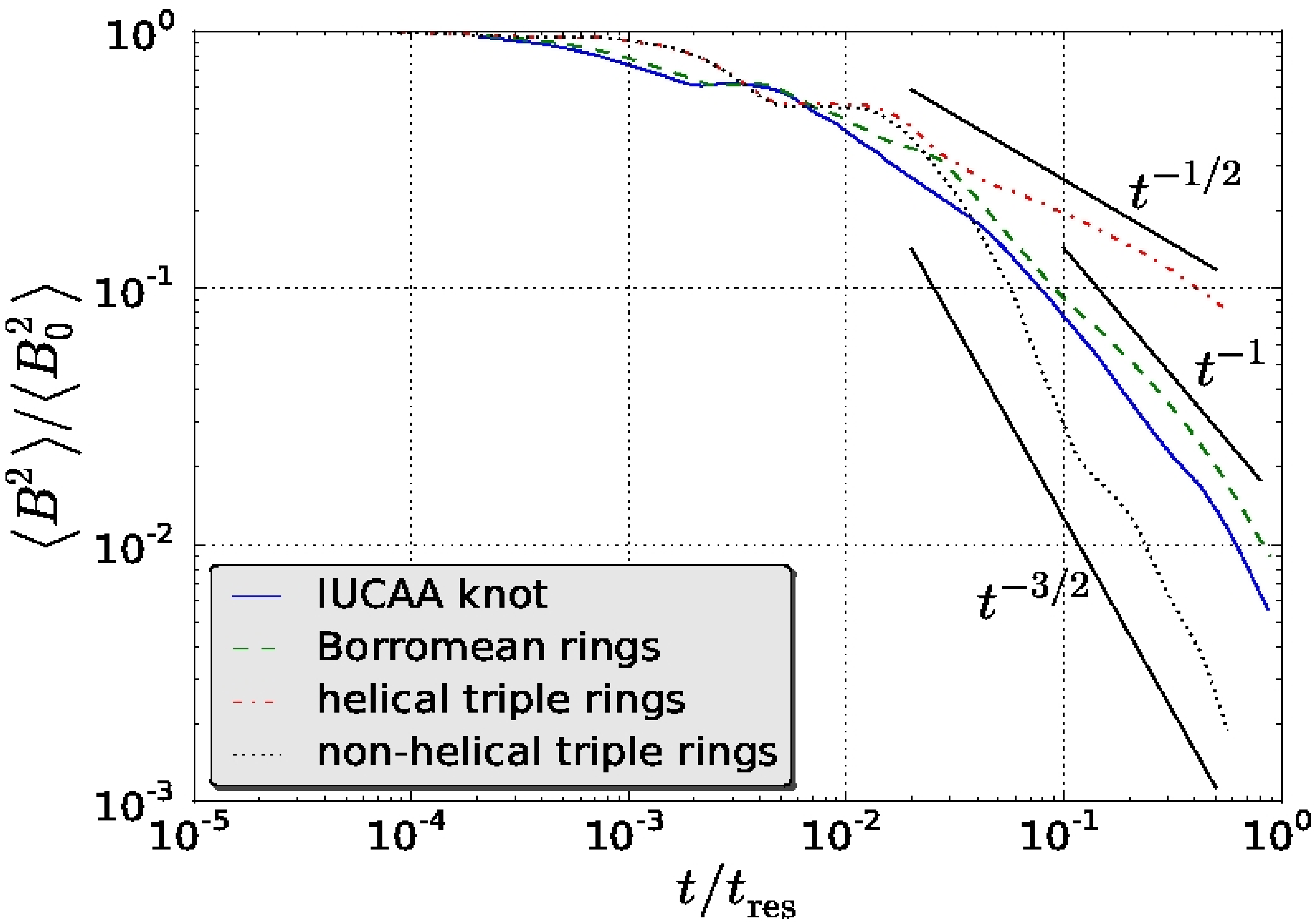
$t = 70$



$t = 78$

3 rings → Twisted ring + interlocked rings → 2 twisted rings

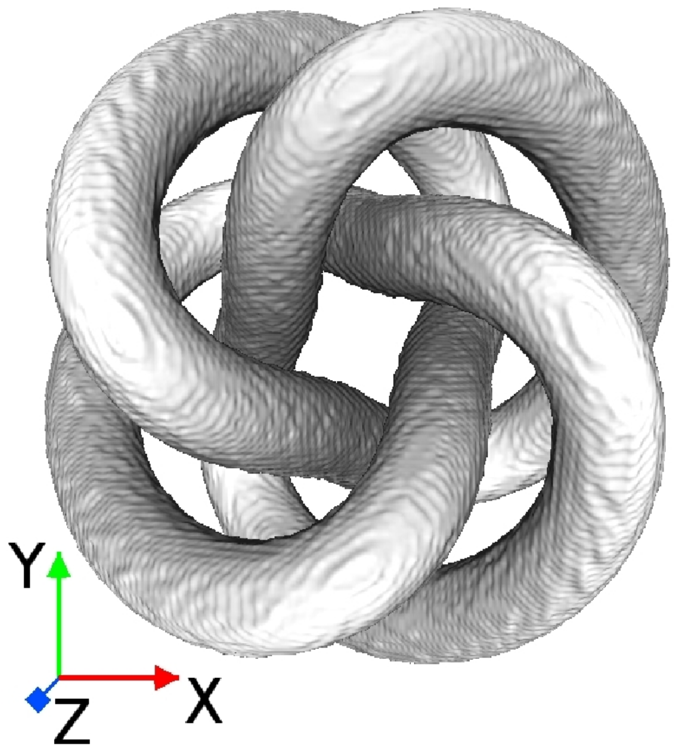
Magnetic Energy Decay



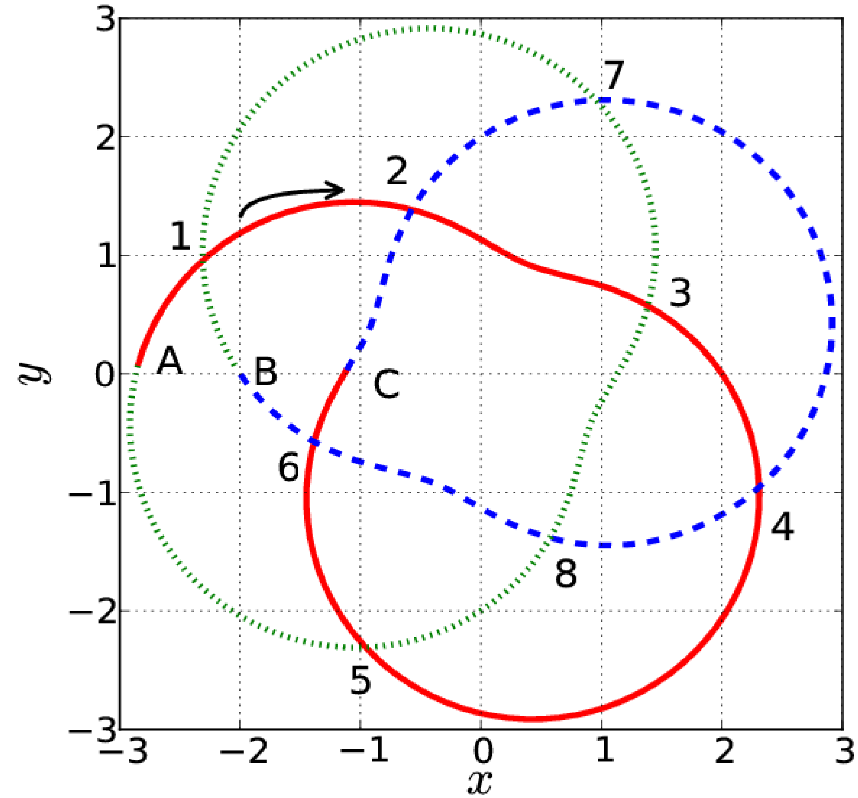
Higher order invariants?

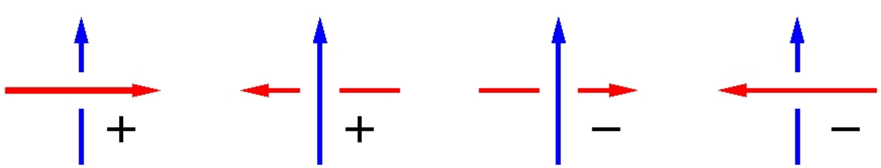
Braid Representation

need $B_z > 0$ → braid representation of knots and links

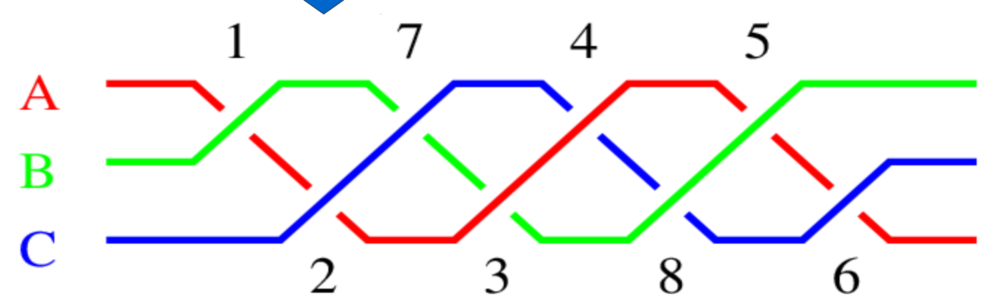


4-foil knot





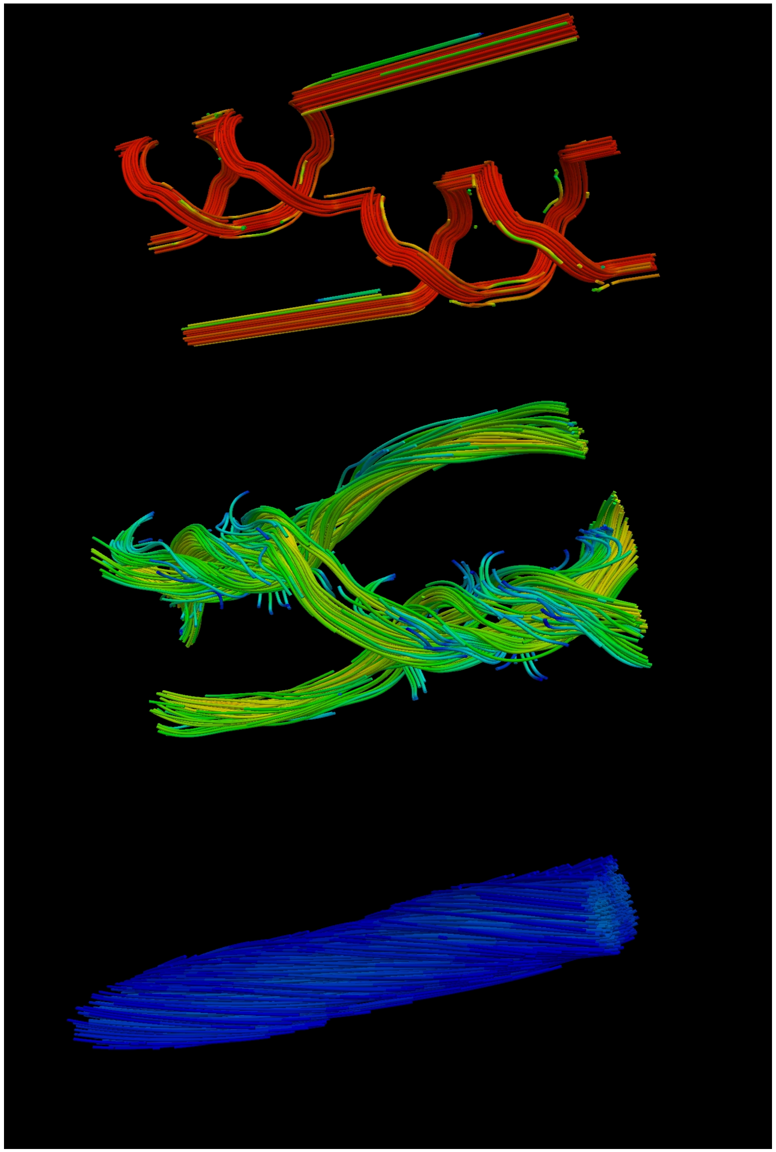
$$n_{\text{linking}} = (n_+ - n_-)/2$$



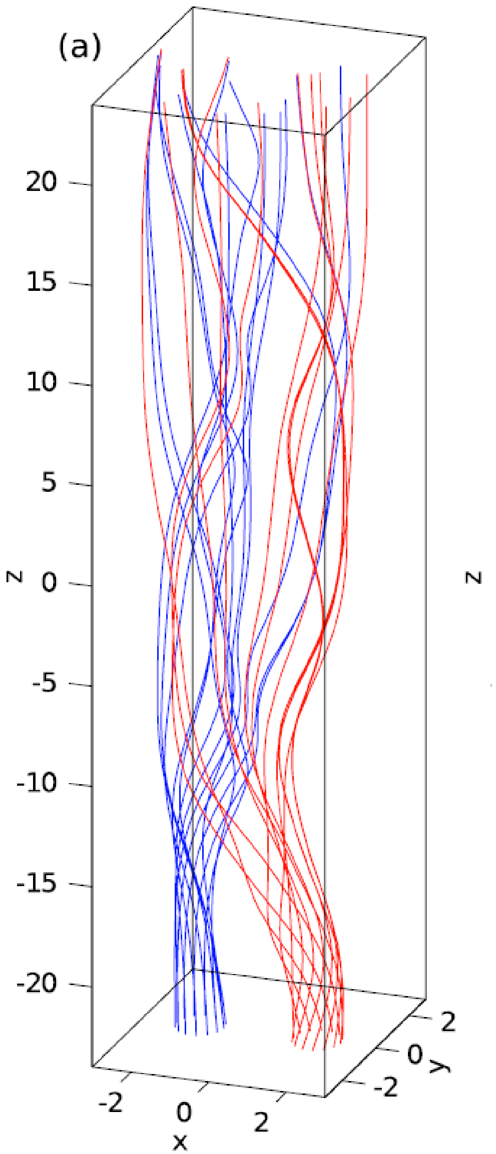
Magnetic Braid Configurations

AAA (trefoil knot)

AABB (Borromean rings)



Fixed Point Index



Yeates et al. 2011a

Trace magnetic field lines from z_0 to z.

mapping: $(x, y) \to \mathbf{F}_z(x, y)$

Color coding:

Compare (x, y) with $\mathbf{F}_1(x, y)$:

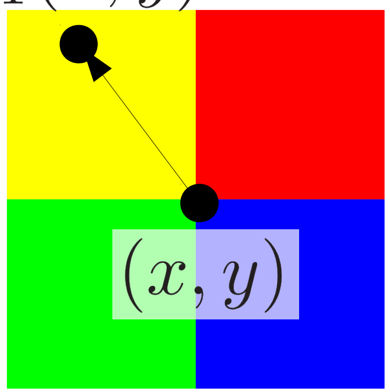
$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y > y$ → red

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y > y$ → yellow

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y < y$ → green

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y < y$ → blue

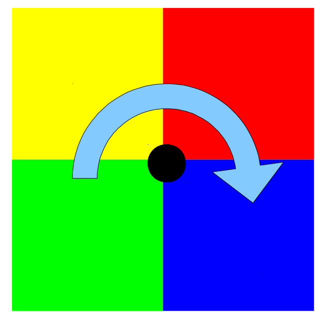
$\mathbf{F}_1(x, y)$



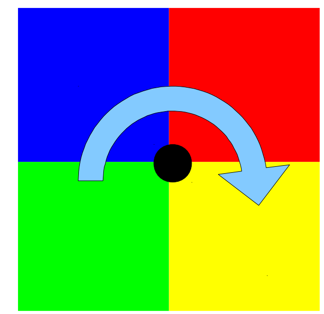
Fixed Point Index

fixed points: $\mathbf{F}_1(x,y) = (x,y)$

Sign t_i of fixed point i :

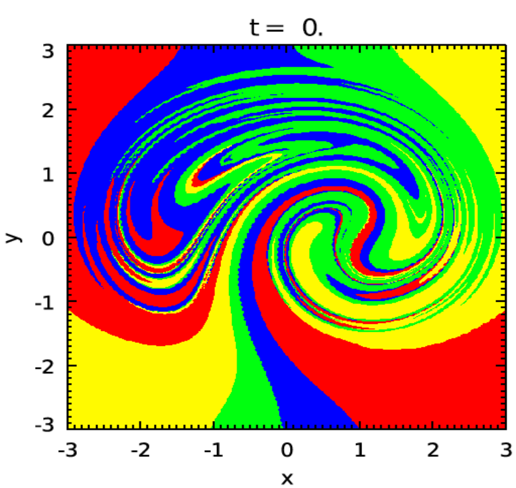


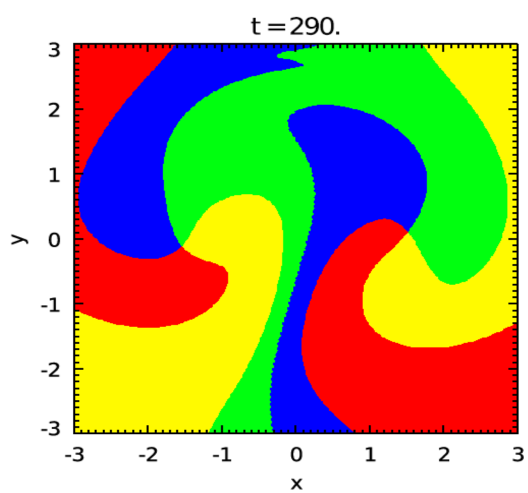
$t_i = +1$



$t_i = -1$

Fixed point index: $T = \sum_i t_i$ conserved for $\lim \eta \to 0$





Taylor state is not reached
$\rightarrow$ T is additional constraint

Magnetic Reconnection Rate

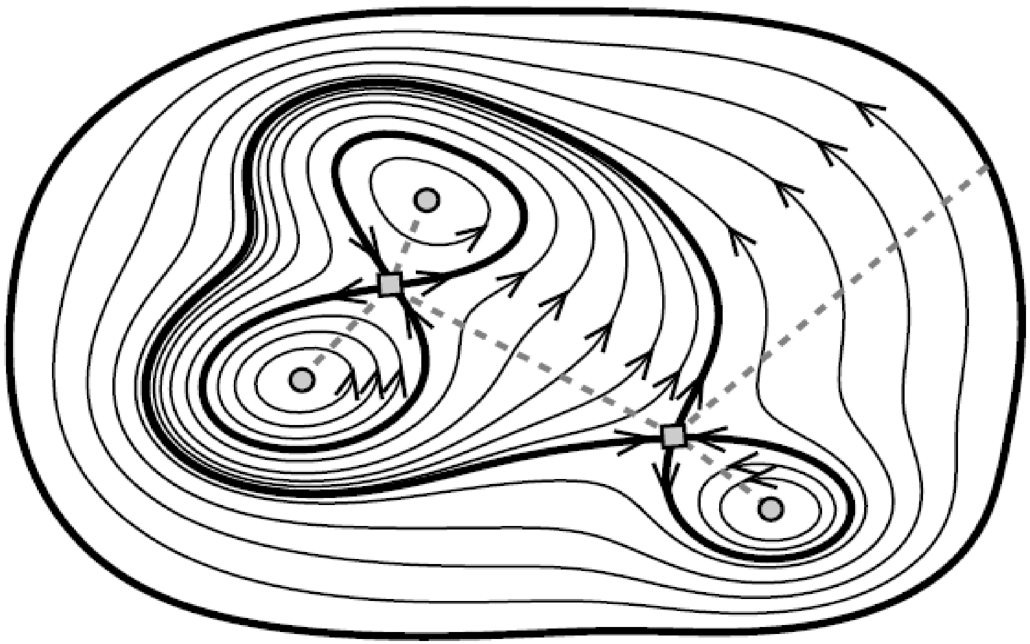
Classic: look for local maxima of $\int \boldsymbol{E} \cdot \boldsymbol{B}$

Partition fluxes 2D:
(Yeates, Hornig 2011b)

$$\boldsymbol{B} = \boldsymbol{\nabla} \times (A\boldsymbol{e}_z)$$

Reconnection rate = magnetic flux through boundaries (spearatrices):

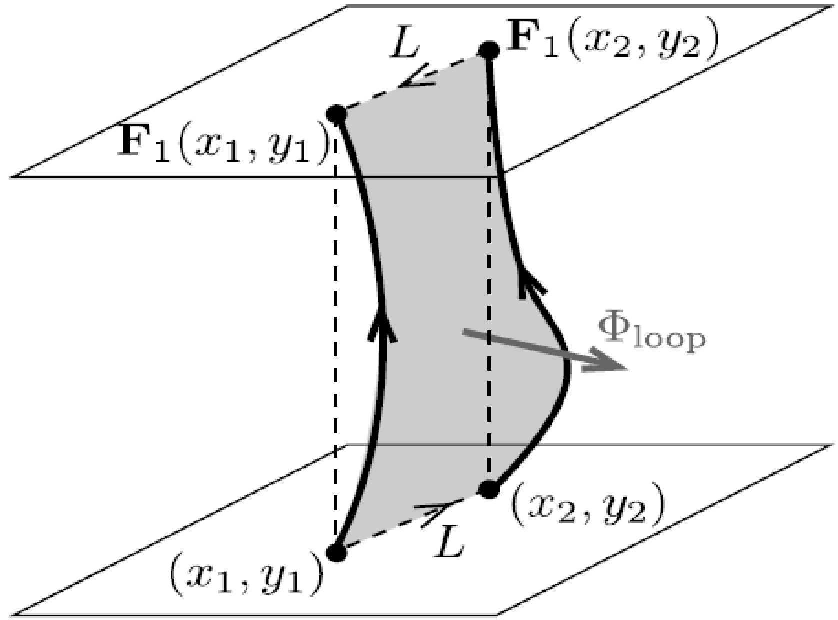
$$\Delta\phi = \sum_i \left| \frac{\mathrm{d}A(\boldsymbol{h}_i)}{\mathrm{d}t} \right|$$



2D Magnetic field.
Thick lines: separatrices.
(Yeates, Hornig 2011b)

Magnetic Reconnection Rate

Partition reconnection rate 3D:
Yeates, Hornig 2011b



Generalized flux function (curly A):

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \boldsymbol{A} \cdot \boldsymbol{B}/B_z \, \mathrm{d}z$$

$$\phi = \int_S \boldsymbol{\nabla} \times \boldsymbol{A} \cdot \, \mathrm{d}\boldsymbol{s} = \int_C \boldsymbol{A} \cdot \, \mathrm{d}\boldsymbol{l}$$

$$\frac{\partial \mathcal{A}}{\partial t} + \boldsymbol{U} \cdot \boldsymbol{\nabla} \mathcal{A} = 0$$

invariant in ideal MHD

Fixed points: $\mathbf{F}_1(x_i, y_i) = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$

Reconnection rate:

$$\Delta\phi = \sum_i \left| \frac{\mathrm{d}\mathcal{A}(\boldsymbol{h}_i)}{\mathrm{d}t} \right|$$

Summary

- Braided magnetic fields are observed in the universe.
- Braiding increases stability through the *realizability condition*.
- Turbulent magnetic field decay is restricted by magnetic helicity.
- Knots and links can be represented as braids.
- Fixed point index as additional constraint in relaxation.
- 'Curly A' as measure for the reconnection rate.

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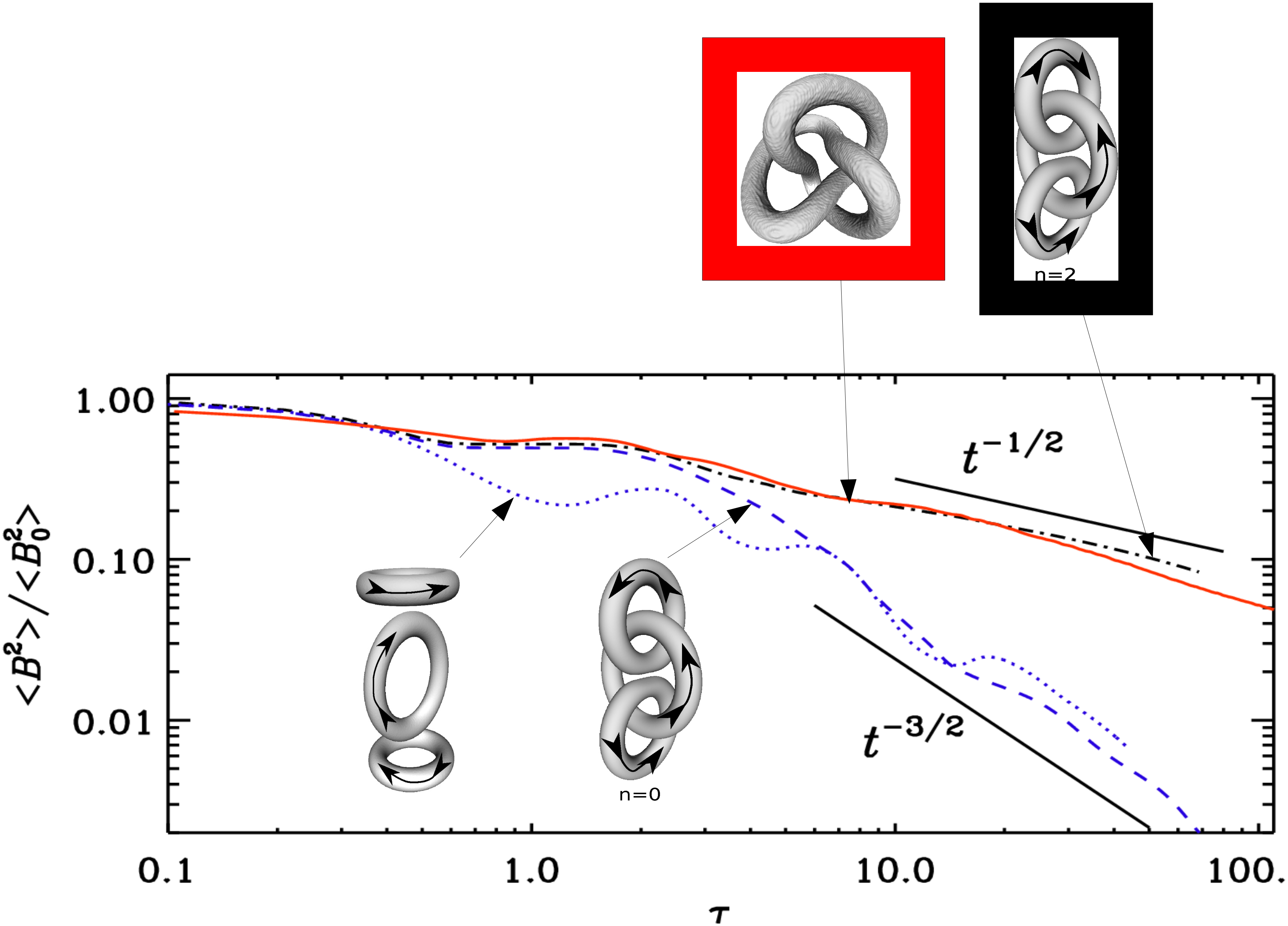
Magnetic energy decay



Simulations

- 256^3 mesh point
- Isothermal compressible gas
- Viscous medium
- Periodic boundaries

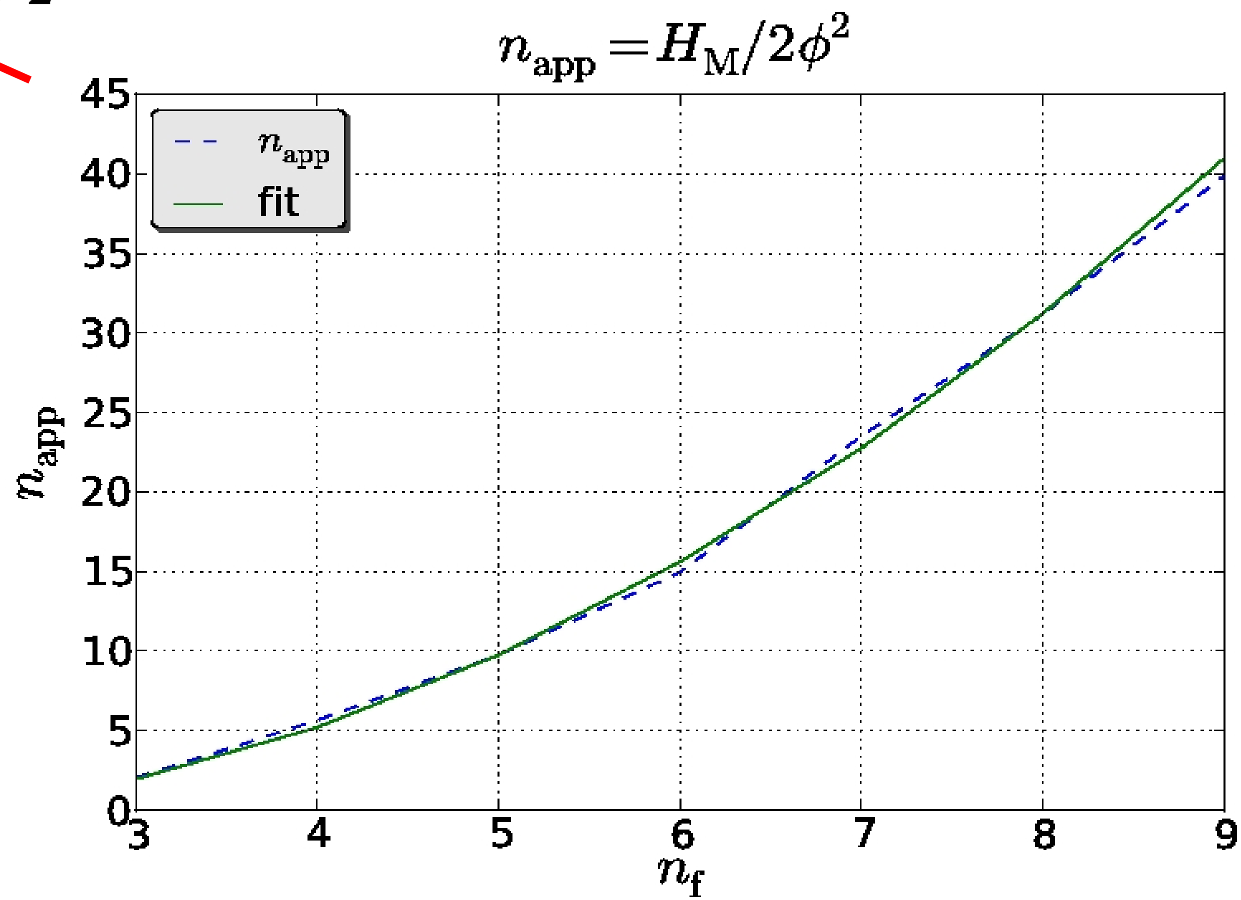
$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \eta \nabla^2 \mathbf{A}$$

$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \boldsymbol{\nabla} \ln \rho + \mathbf{J} \times \mathbf{B}/\rho + \mathbf{F}_{\mathrm{visc}}$$

$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{U}$$

N-foil Knots

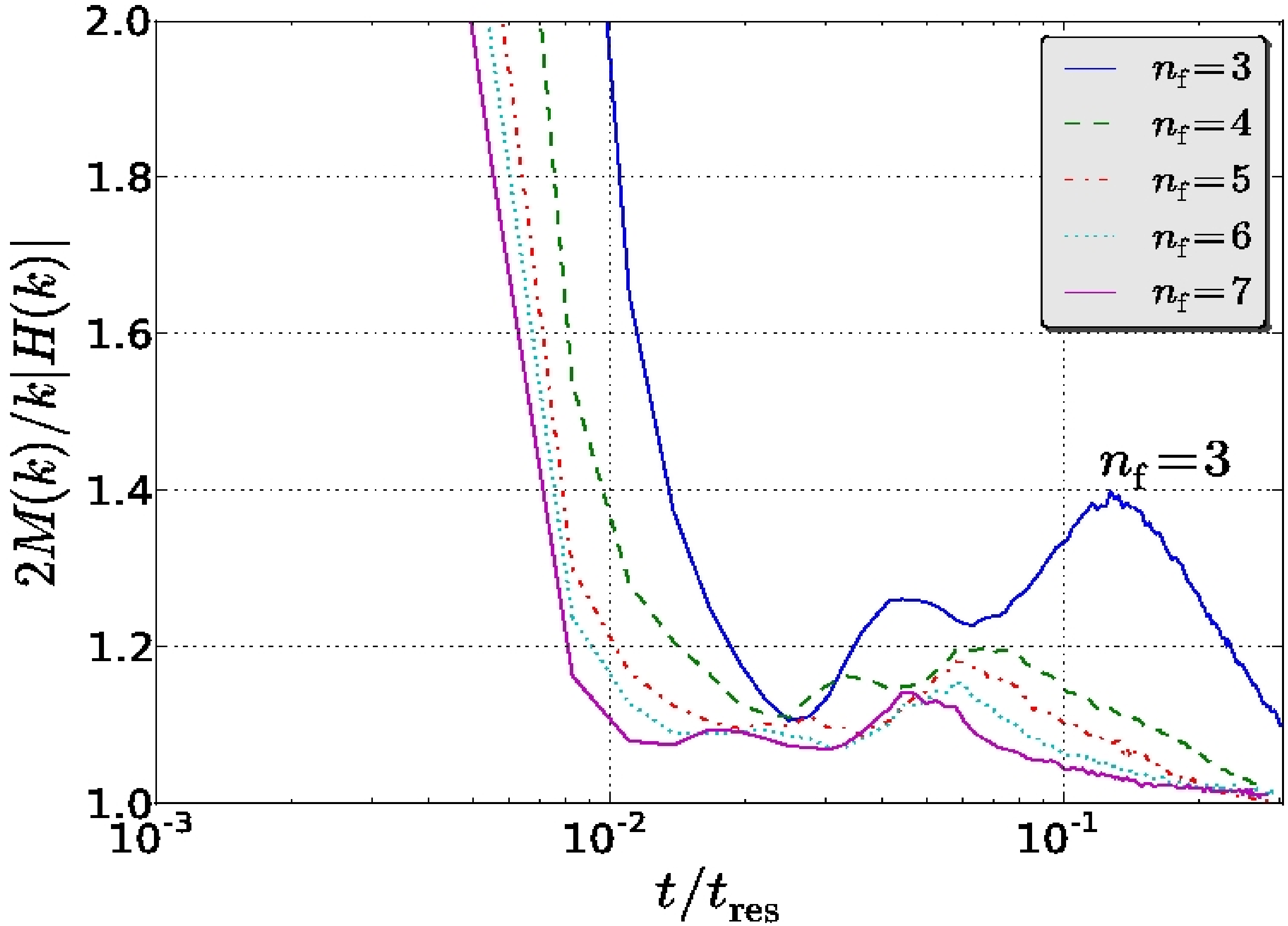
~~$H_{\mathrm{M}} = 2n\phi_1\phi_2$~~



$$H_{\mathrm{M}} = (n_{\mathrm{f}} - 2)n_{\mathrm{f}}\phi^2/2$$

N-foil Knots

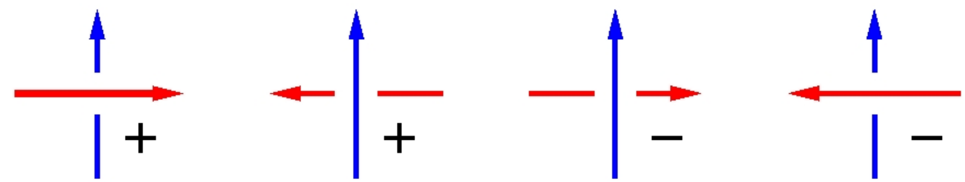
$2M(k)/(|H(k)|k)$



$k = 2$

Realizability condition more important for high $n_{\rm f}$.

Linking Number

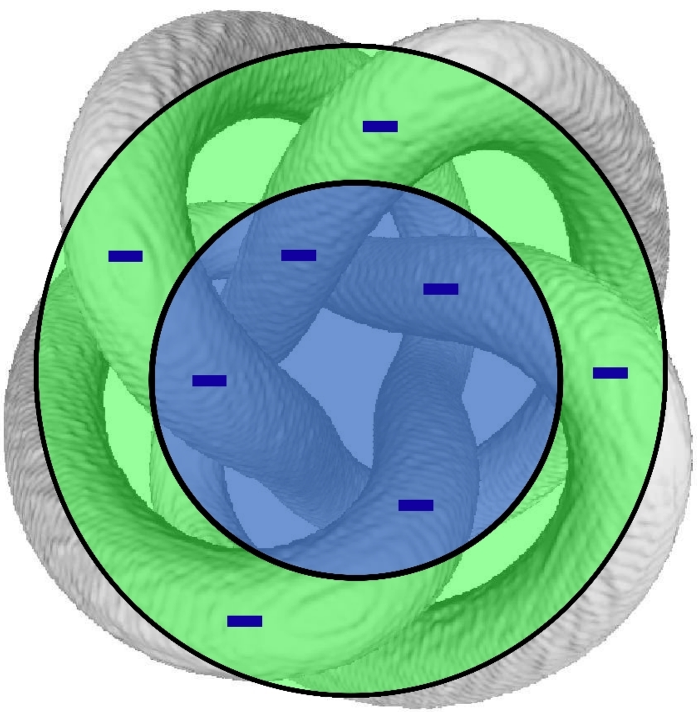


$$n_{\rm linking} = (n_+ - n_-)/2$$

Number of crossings increases like $n_{\rm f}^2$

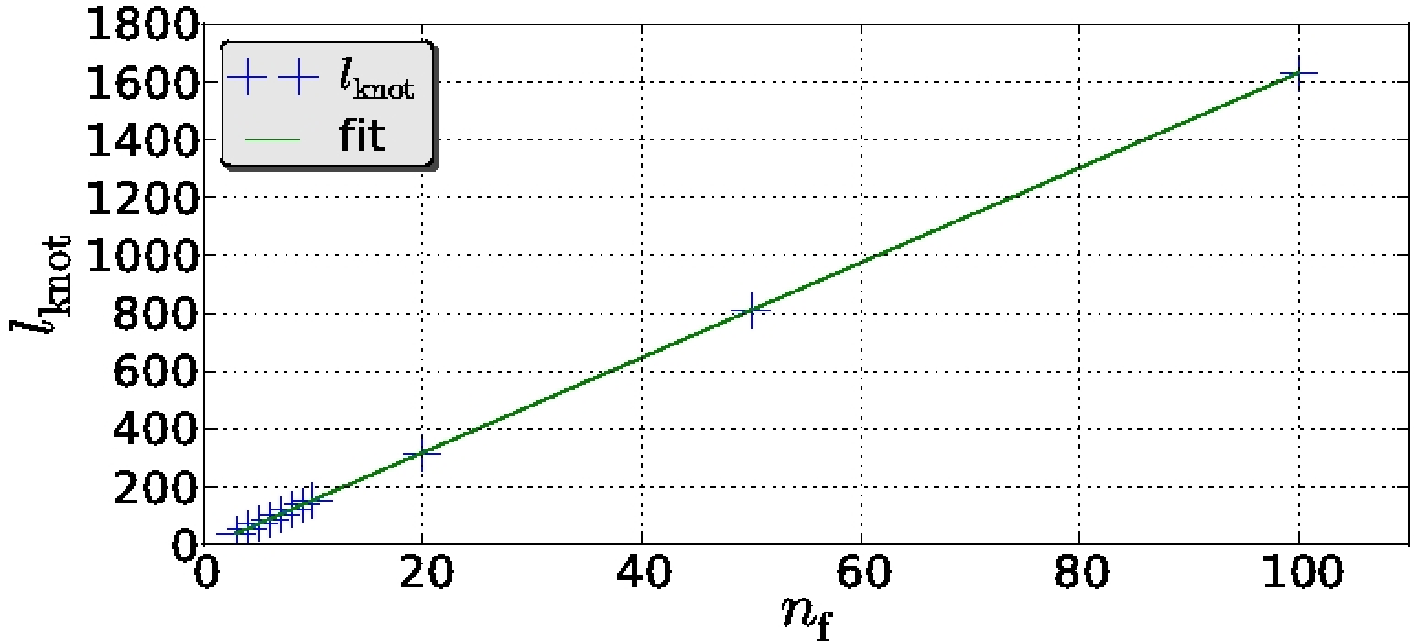
$$H_{\rm M} \propto n_{\rm linking}$$

$$H_{\rm M} \propto n_{\rm f}^2$$



Sign of the crossings for the 4-foil knot

Helicity vs. Energy



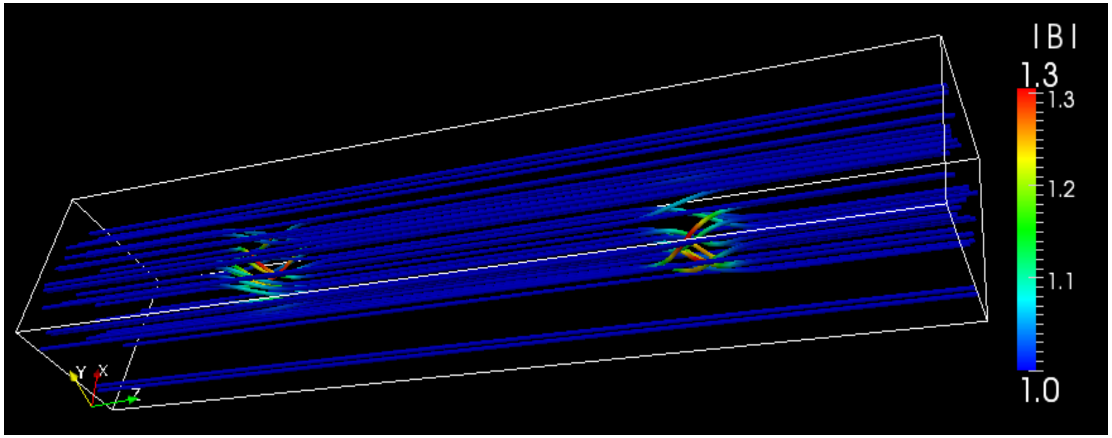
$$E_{\mathrm{M}} \propto l_{\mathrm{knot}} \propto n_{\mathrm{f}}$$

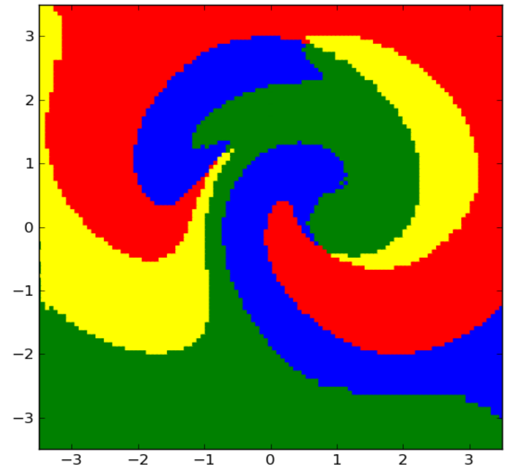
$$H_{\mathrm{M}} \propto n_{\mathrm{f}}^2$$

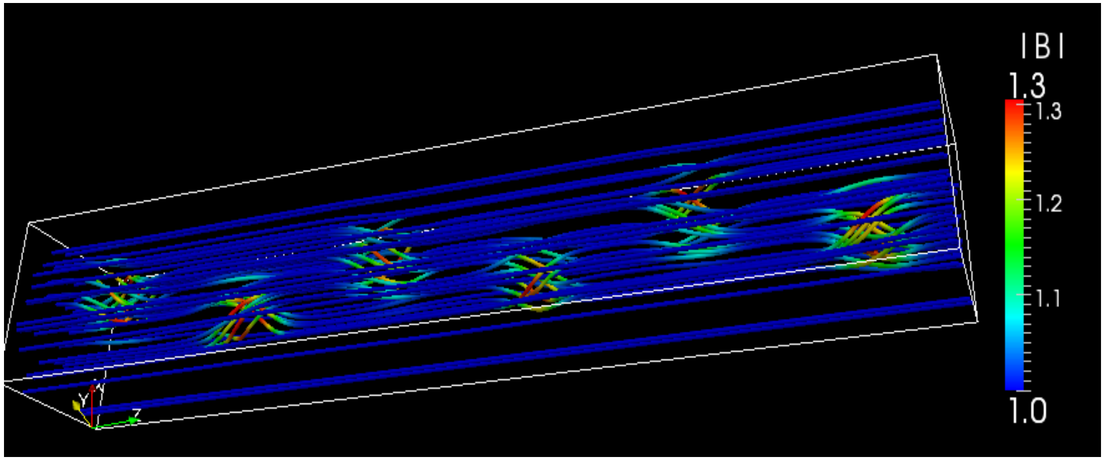
Knot is more strongly packed with increasing .

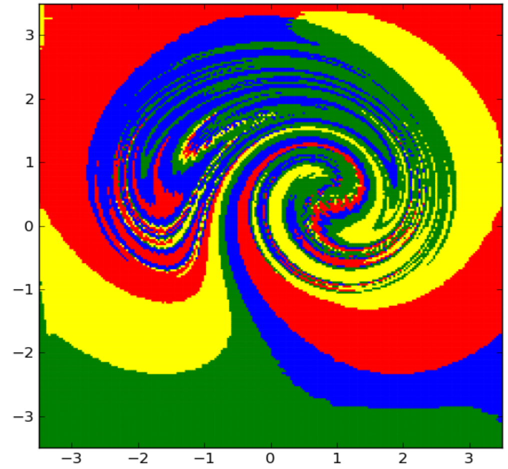
Magnetic energy is closer to its lower limit for high .

Field Line Tracing









Generalized flux function:

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

Reconnection rate:

$$\sum_i \frac{\mathrm{d}\mathcal{A}(\mathbf{x}_i)}{\mathrm{d}t}$$