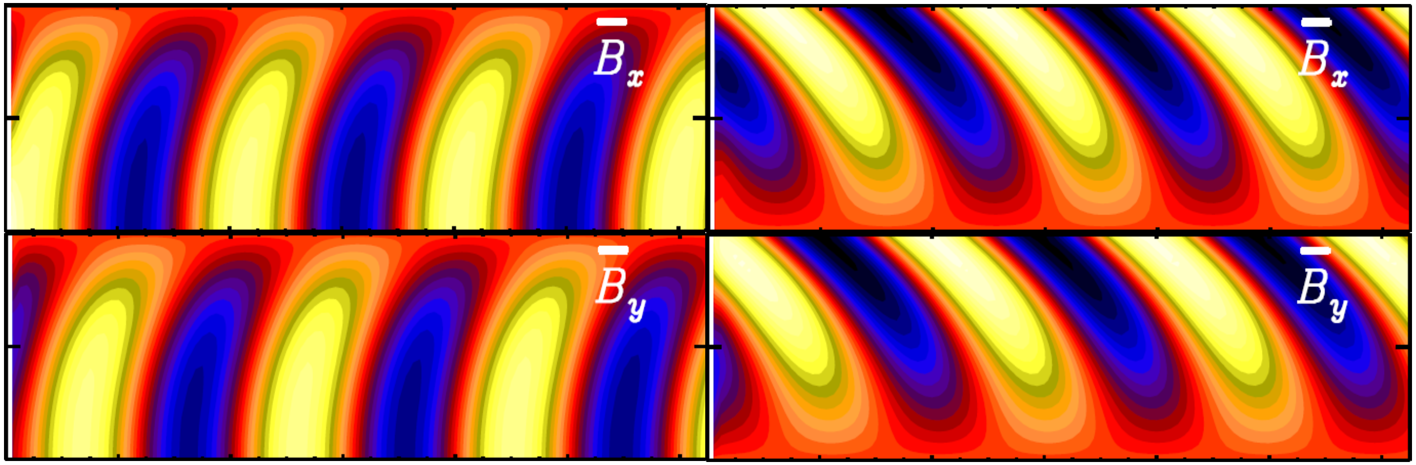
Effects of magnetic helicity in turbulent dynamos

Simon Candelaresi

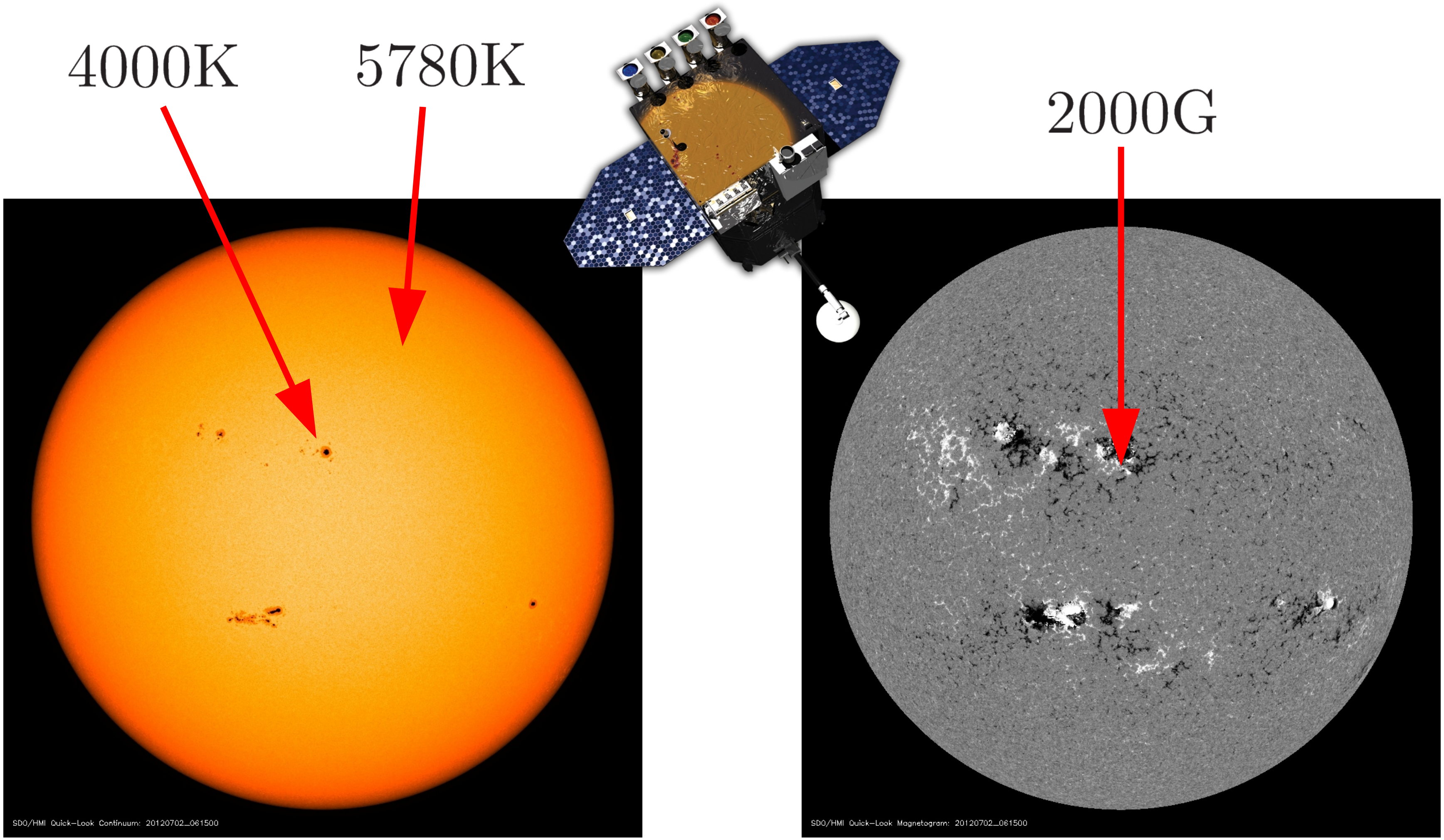


Outline

- Observations of sunspots and magnetic fields.
- Dynamo mechanism.
- Mean-field model.
- Alpha-effect and alpha-quenching.
- Magnetic helicity fluxes.

- Measure of topology.
- actual linking vs. magnetic helicity
- Fixed point index.

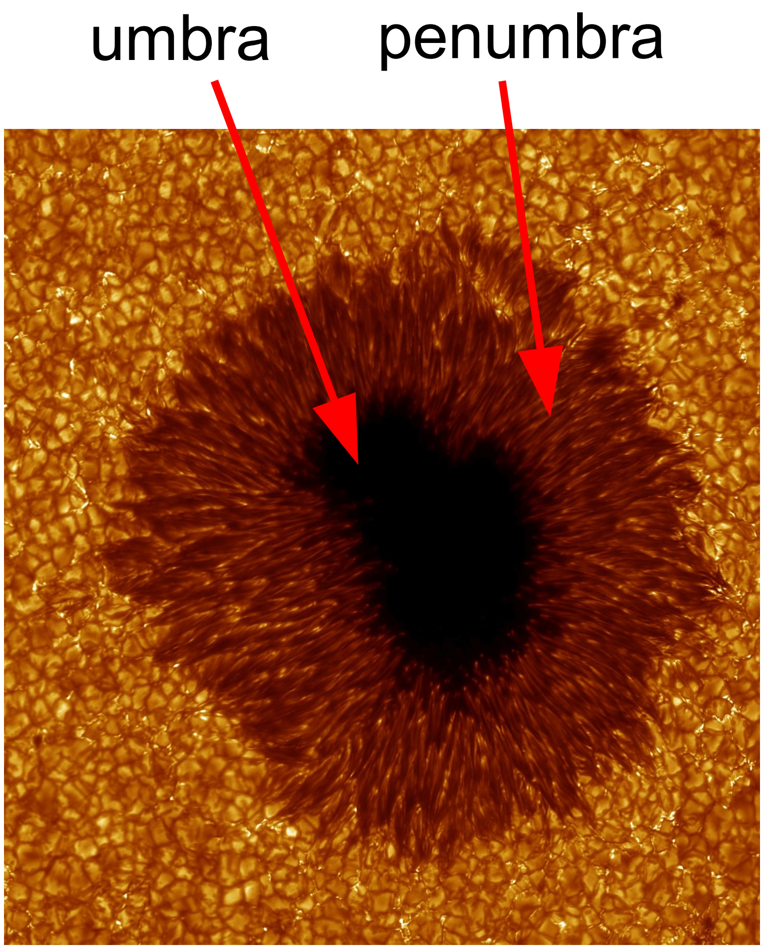
Solar Dynamics Observatory (SDO)



2nd July 2012, Intensity

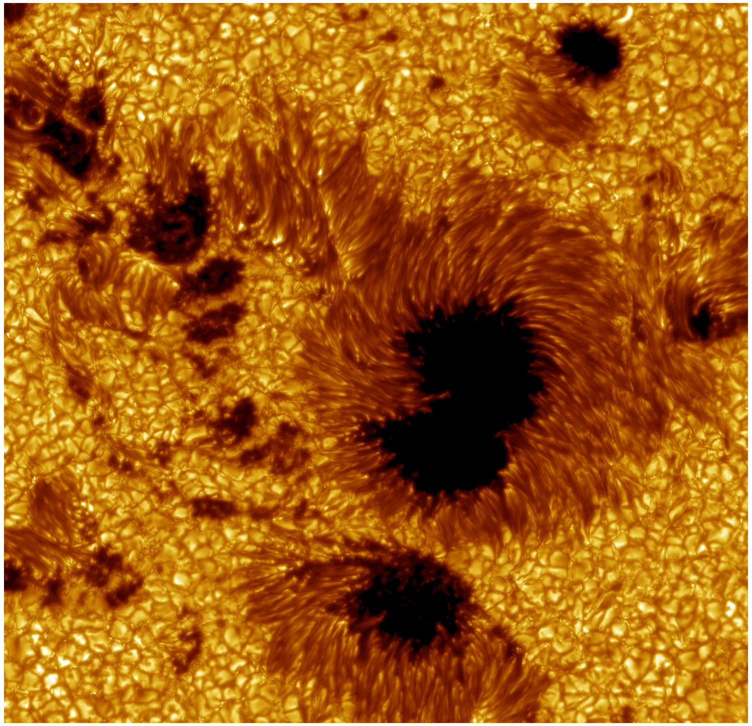
2nd July 2012, Magnetogram

Swedish Solar Telescope (SST)



La Palma

(Göran Scharmer)

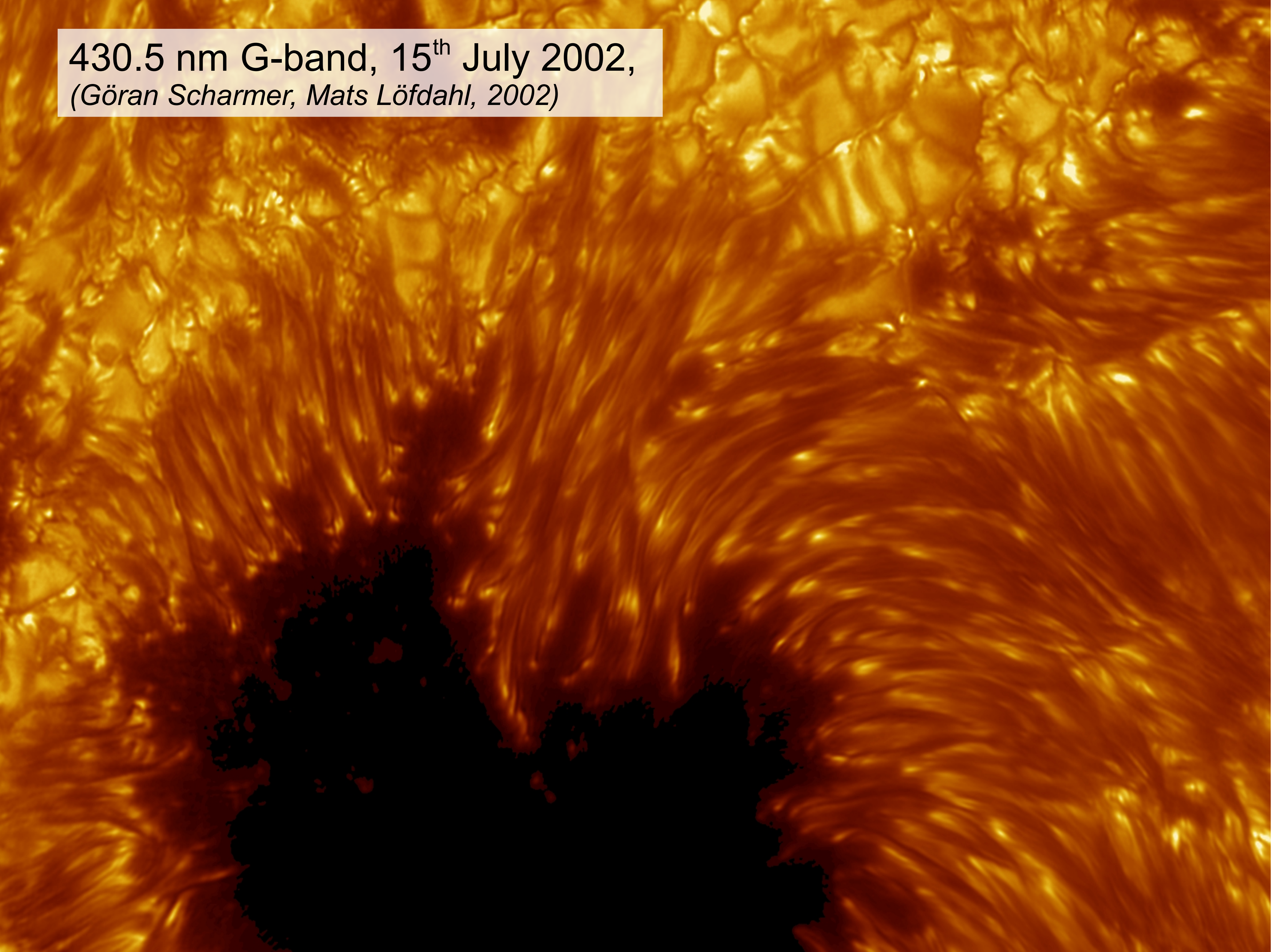


430.5 nm (G-band), 3rd July 2003,

(Dan Kiselman, Mats Löfdahl, 2003)

487.7 nm, 15th July 2002,

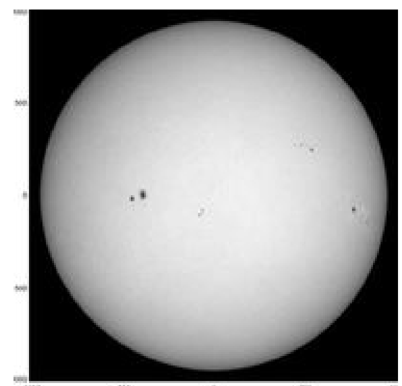
(Göran Scharmer, Mats Löfdahl, 2002)

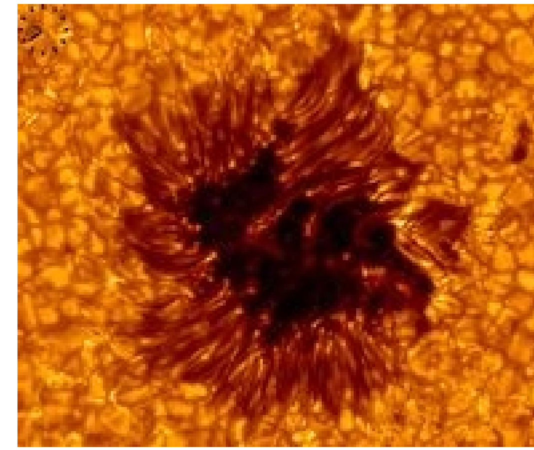

430.5 nm G-band, 15th July 2002,
(Göran Scharmer, Mats Löfdahl, 2002)

Swedish Solar Telescope (SST)

1h quiet Sun, 656.3 nm,
18th June 2006,
(Luc Rouppe van der Voort, Oslo, 2006)

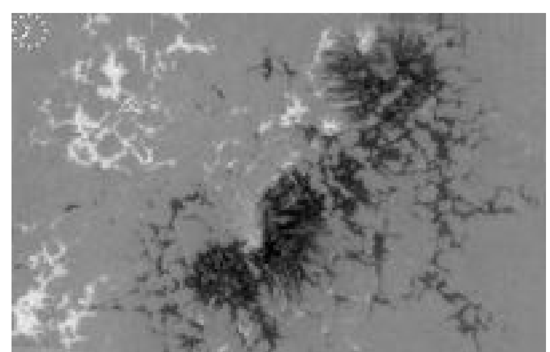
Zoom from SOHO/MDI field of view to SST resolution, August 2004,
(Michiel van Noort, Luc Rouppe van der Voort, Mats Carlsson, Oslo, 2004)



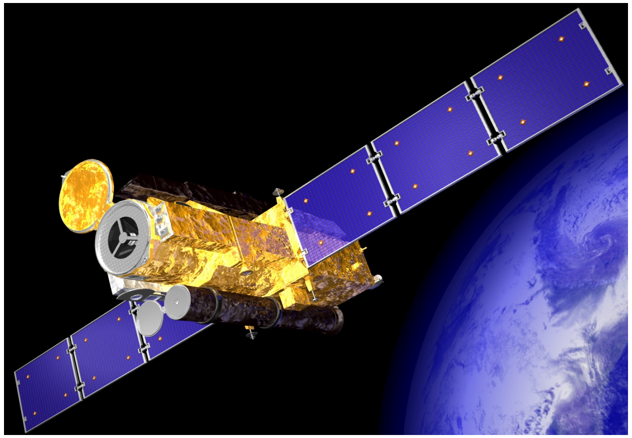


Sunspot 41 min, 430.5 nm G-band, 20th August 2004,
(Michiel van Noort and Luc Rouppe van der Voort, Oslo, 2004)

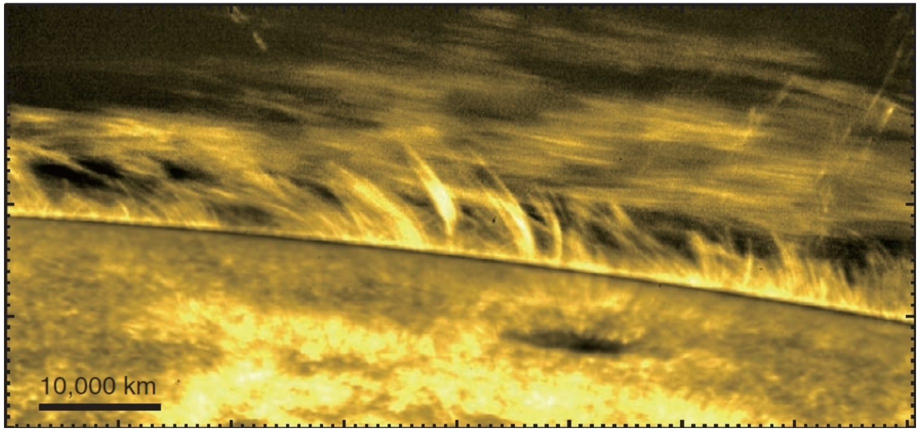
Sunspot group magnetogram, 21st August 2004,
(Michiel van Noort and Luc Rouppe van der Voort, Oslo, 2004)



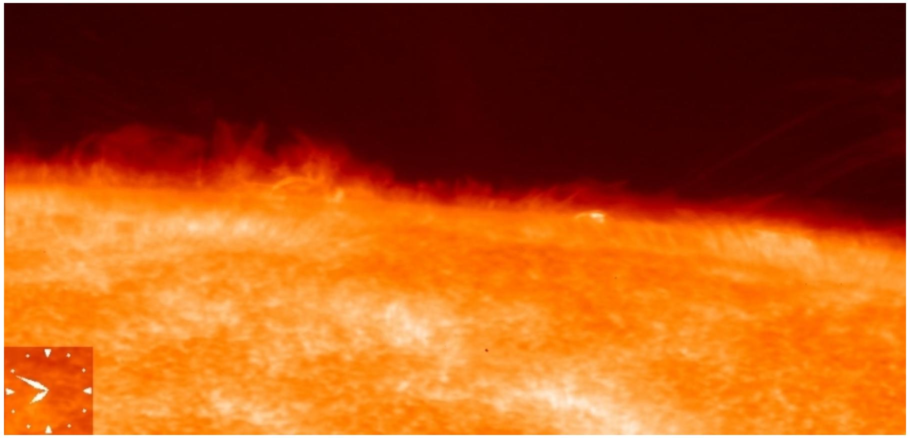
Hinode ひので (Solar-B)



(JAXA)



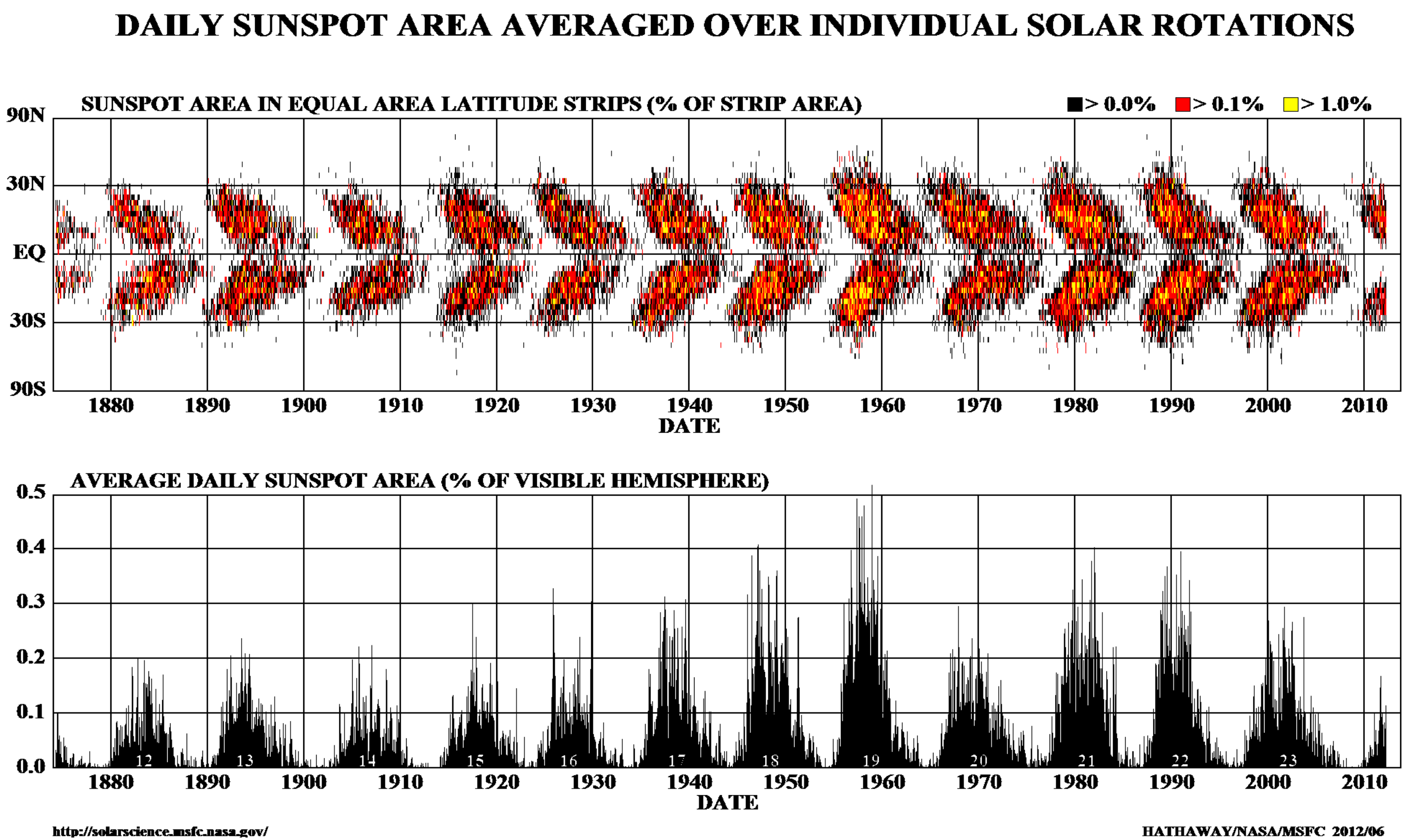
Solar prominence,
9^{th} November 2006,
(Okamoto, T.J. et al., 2007)



Eruption observed in Ca II H (397nm) above a Sun spot,
http://solarb.msfc.nasa.gov/news/movies.html

Solar Magnetic Field

11 year cycle

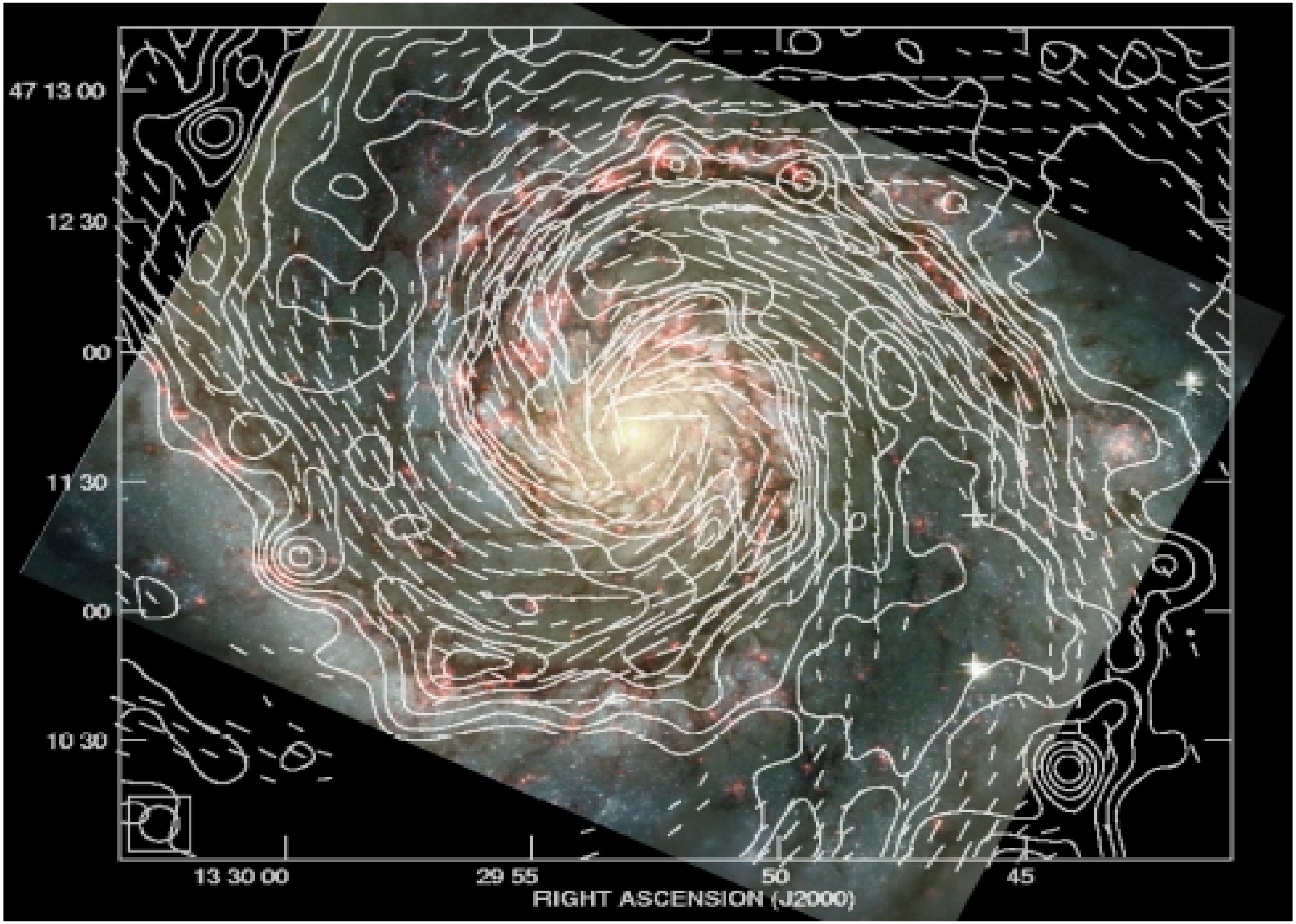


蝶形図

dynamo working

(Hathaway/NASA)

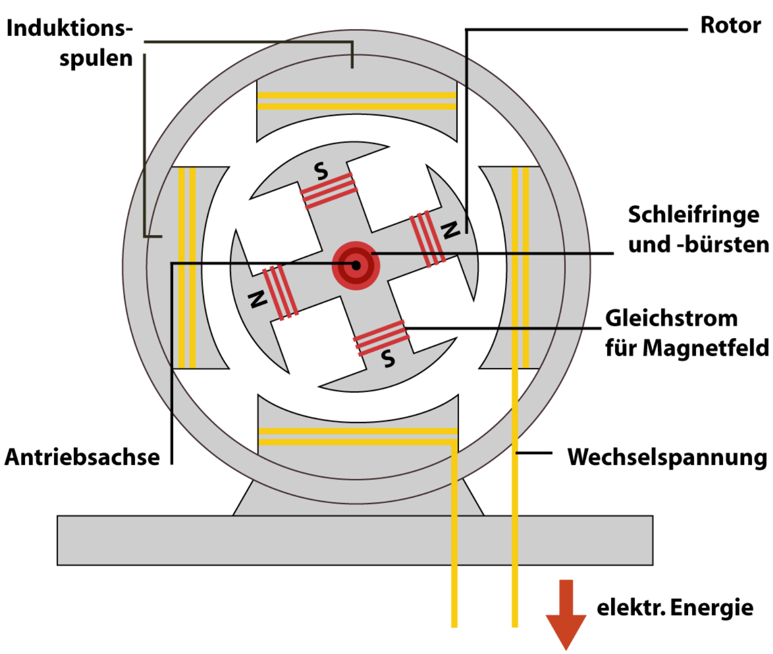
Galactic Magnetic Fields



Galaxy M51, radio + optical
(Fletcher et al. 2011)

Dynamo Effect

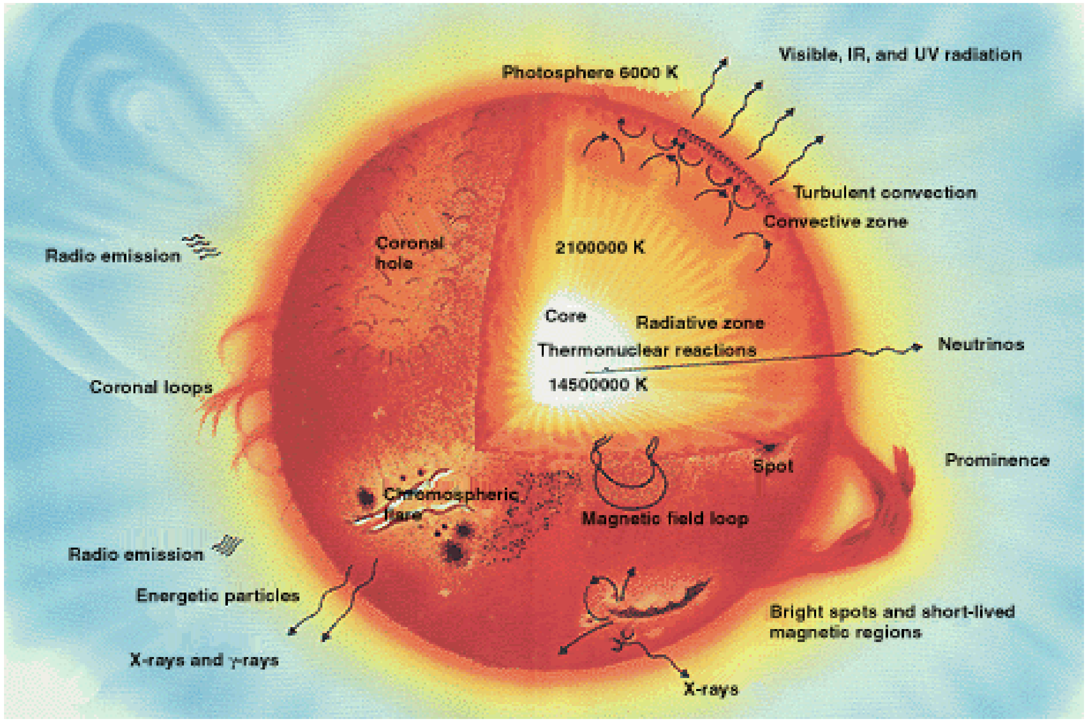
kinetic motion → induction
→ electric energy



electric power generator
(Wikipedia, user: Kuntoff, 2005)

$$\partial_t \boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{E}$$

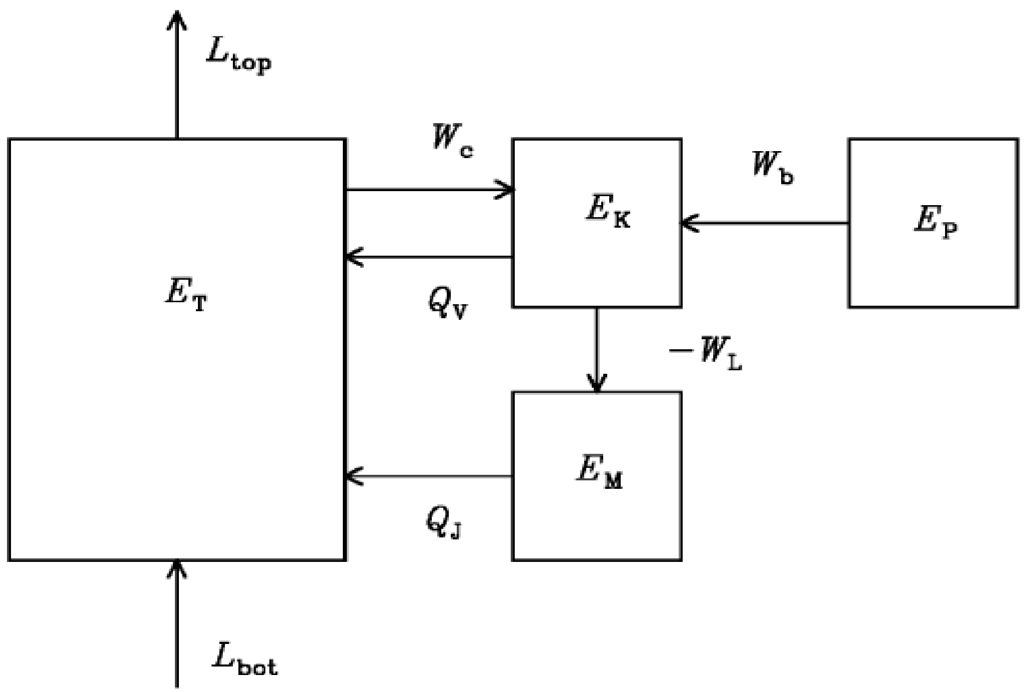
turbulent motion → induction
→ magnetic energy



Solar model
(NASA)

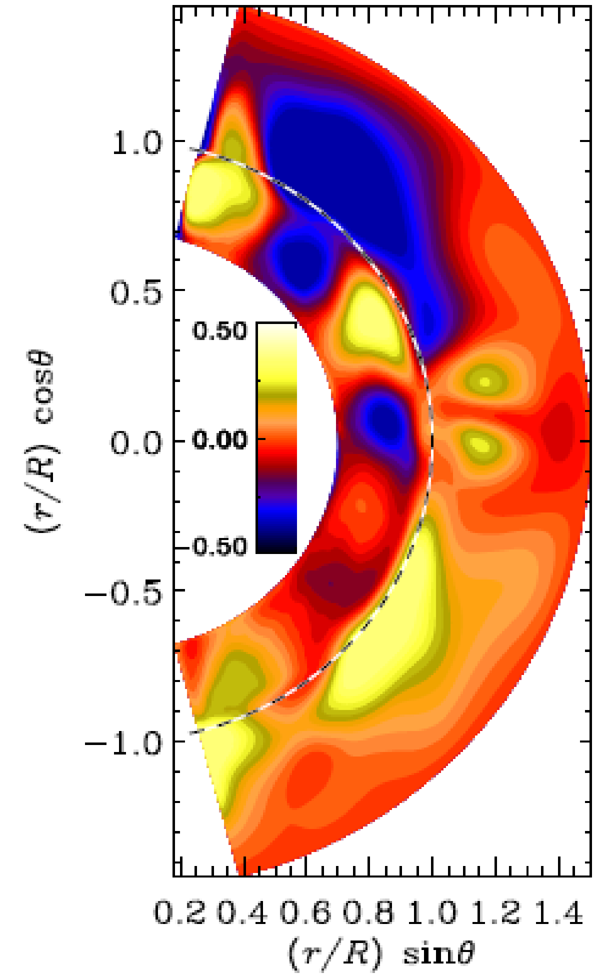
$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{B}$$

Turbulent Dynamo Schematics



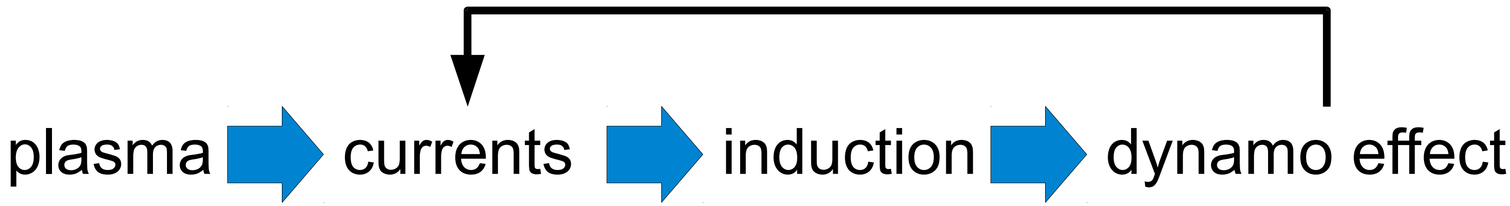
Energy budget for a dynamo.
(Brandenburg et al., 1996)

$E_T, E_K, E_M, E_P =$
thermal, kinetic, magnetic and potential energy



$\langle \overline{B}_\phi \rangle_t$ for a convection driven dynamo.
(Warnecke et al., 2012)

Dynamo Mechanism



Equations of **magnetohydrodynamics** (MHD):

Induction equation: $$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B} - \eta \boldsymbol{J})$$

Momentum equation: $$\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \boldsymbol{\nabla} \ln \rho + \boldsymbol{J} \times \boldsymbol{B}/\rho + \boldsymbol{F}_{\mathrm{visc}}$$

Continuity equation: $$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \boldsymbol{U}$$

Advective derivative: $$\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{U} \cdot \boldsymbol{\nabla}$$

Mean-Field Formalism

Mean-field decomposition: $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$

Reynolds rules: $\overline{\boldsymbol{B}_1 + \boldsymbol{B}_2} = \overline{\boldsymbol{B}}_1 + \overline{\boldsymbol{B}}_2, \quad \overline{\overline{\boldsymbol{B}}} = \overline{\boldsymbol{B}}, \quad \overline{\boldsymbol{b}} = 0$

$$\overline{\partial_\mu \boldsymbol{B}} = \partial_\mu \overline{\boldsymbol{B}}, \quad \mu = 0, 1, 2, 3$$

Mean-field induction equations:

$$\partial_t \overline{\boldsymbol{B}} = \eta \nabla^2 \overline{\boldsymbol{B}} + \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{\mathcal{E}}})$$

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{G}) + \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

Electromotive force (emf): $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$

$$\boldsymbol{G} = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

need closure ➡ express $\overline{\boldsymbol{\mathcal{E}}}$ in terms of the mean fields:

$$\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{\mathcal{E}}}(\overline{\boldsymbol{U}}, \overline{\boldsymbol{B}}, ...)$$

Mofatt, H. K., 1978, Krause and Raedler, 1980

Electromotive Force

The EMF is assumed to be linear and homogeneous in $\overline{\boldsymbol{B}}$.

$$\mathcal{E}_i(x,t) = \mathcal{E}_i^{(0)}(x,t) + \int\int_\alpha K_{ij}(x,x',t,t')\overline{B}_j(x-x',t-t')\ \mathrm{d}^3x'\ \mathrm{d}t'$$

Taylor expansion:

$$\overline{B}_j(x',t) = \overline{B}_j(x,t) + (x'_k - x_k)\frac{\partial \overline{B}_j(x,t)}{\partial x_k} + \ldots$$

Assume local and instantaneous dependence of $\overline{\boldsymbol{\mathcal{E}}}$ on $\overline{\boldsymbol{B}}$.

$$\overline{\mathcal{E}}_i = \alpha_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k} + \ldots$$

For a turbulent system without preferred direction, i.e. $\boldsymbol{U} = 0$:

$$\overline{\boldsymbol{\mathcal{E}}} = \alpha\overline{\boldsymbol{B}} - \eta_{\mathrm{t}}\nabla\times\overline{\boldsymbol{B}}$$

$$\partial_t\overline{\boldsymbol{B}} = \alpha\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} + \eta_{\mathrm{T}}\boldsymbol{\nabla}^2\overline{\boldsymbol{B}}$$

Nonlinear Alpha-Effect

Helical forcing: $\alpha = \alpha_\mathrm{K} = -\tau\overline{\boldsymbol{\omega}\cdot\boldsymbol{u}}/3$

Back reaction of $\overline{\boldsymbol{B}}$ on α

➡ Correction: α should depend on $\overline{\boldsymbol{B}}$

$$\alpha = \alpha_\mathrm{K}(1 - \overline{\boldsymbol{B}}^2/B_\mathrm{eq}^2) \quad (\overline{\boldsymbol{B}}^2 \ll B_\mathrm{eq}^2)$$ *(Roberts 1975)*

Algebraic (conventional) quenching:

$$\alpha = \frac{\alpha_\mathrm{K}}{1 + \overline{\boldsymbol{B}}^2/B_\mathrm{eq}^2}$$ *(Ivanova 1977)*

Catastrophic quenching (fit):

$$\alpha = \frac{\alpha_\mathrm{K}}{1 + R_\mathrm{m}\overline{\boldsymbol{B}}^2/B_\mathrm{eq}^2}$$

(Vainshtein 1992)

Sun:	$R_\mathrm{m} = 10^9$
Galaxies:	$R_\mathrm{m} = 10^{18}$

Alpha-Effect

α effect: $\alpha = \alpha_\mathrm{K} + \alpha_\mathrm{M}$ (magnetic helicity conservation)

$$\alpha_\mathrm{K} = -\tau\overline{\boldsymbol{\omega}\cdot\boldsymbol{u}}/3$$

$$\alpha_\mathrm{M} = \tau\overline{\boldsymbol{j}\cdot\boldsymbol{b}}/(3\overline{\rho}) = \tau k^2\overline{\boldsymbol{a}\cdot\boldsymbol{b}}/(3\overline{\rho}) = \overline{h}_\mathrm{f}$$

(Pouquet et al. 1976)

helically driven dynamo $\overline{h}_\mathrm{K,f} = \overline{\boldsymbol{\omega}\cdot\boldsymbol{u}}$

- production of magnetic helicity $\overline{h}_\mathrm{M,f} = \overline{\boldsymbol{a}\cdot\boldsymbol{b}}$
- total magnetic helicity conservation $\overline{h}_\mathrm{M,m} = \overline{\boldsymbol{A}}\cdot\overline{\boldsymbol{B}}$

$\overline{\boldsymbol{a}\cdot\boldsymbol{b}}$ works against dynamo: $E_\mathrm{M} \propto 1/\mathrm{Re}_\mathrm{M}$ $\mathrm{Re}_\mathrm{M} = \dfrac{UL}{\eta}$

Sun: $\mathrm{Re}_\mathrm{M} = 10^9$ galaxies: $\mathrm{Re}_\mathrm{M} = 10^{18}$

Dynamical Alpha-Quenching

Magnetic helicity evolution:

mean: $$\frac{\partial \overline{h}_{\rm m}}{\partial t} = 2\overline{\boldsymbol{\mathcal{E}}} \cdot \overline{\boldsymbol{B}} - 2\eta\mu_0 \overline{\boldsymbol{J}} \cdot \overline{\boldsymbol{B}} - \boldsymbol{\nabla} \cdot \overline{\boldsymbol{F}}_{\rm m}$$

fluctuating: $$\frac{\partial \overline{h}_{\rm f}}{\partial t} = -2\overline{\boldsymbol{\mathcal{E}}} \cdot \overline{\boldsymbol{B}} - 2\eta\mu_0 \overline{\boldsymbol{j} \cdot \boldsymbol{b}} - \boldsymbol{\nabla} \cdot \overline{\boldsymbol{F}}_{\rm f}$$

$$\alpha_{\rm M} = \overline{h}_{\rm f}$$

$$\frac{\partial \alpha_{\rm M}}{\partial t} = -2\eta_{\rm t} k_{\rm f}^2 \left(\frac{\overline{\boldsymbol{\mathcal{E}}} \cdot \overline{\boldsymbol{B}}}{B_{\rm eq}^2} + \frac{\alpha_{\rm M}}{R_{\rm m}} \right) - \frac{\partial}{\partial z} \overline{\mathcal{F}}_\alpha$$

Magnetic Helicity Fluxes

$$\text{I)}\quad \frac{\partial \alpha_{\rm M}}{\partial t} = -2\eta_{\rm t} k_{\rm f}^2 \left(\frac{\overline{\boldsymbol{\mathcal{E}}}\cdot\overline{\boldsymbol{B}}}{B_{\rm eq}^2} + \frac{\alpha_{\rm M}}{R_{\rm m}} \right) - \frac{\partial}{\partial z}\overline{\mathcal{F}}_\alpha$$

$$\text{II)}\quad \partial_t \overline{\boldsymbol{B}} = \alpha \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} + \eta_{\rm T} \boldsymbol{\nabla}^2 \overline{\boldsymbol{B}}$$

$$\text{III)}\quad \overline{\boldsymbol{\mathcal{E}}} = \alpha \overline{\boldsymbol{B}} - \eta_{\rm t} \nabla \times \overline{\boldsymbol{B}}$$

1D mean-field in z

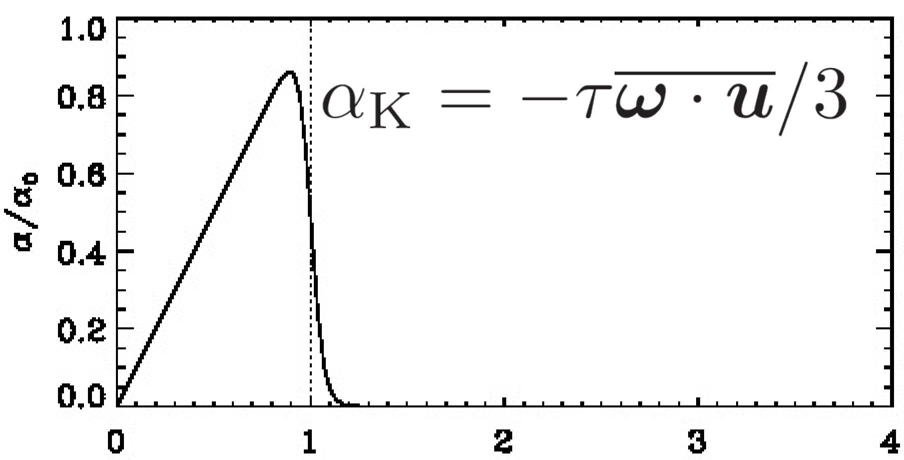
α diffusion

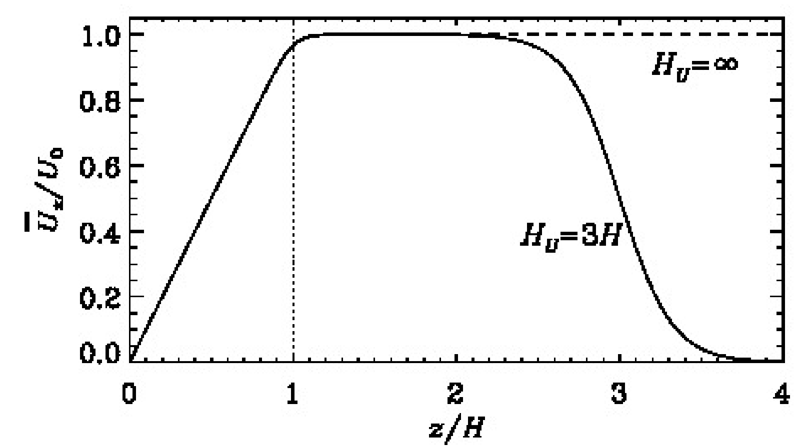
$$\kappa_\alpha \frac{\partial \alpha_{\rm M}}{\partial z}$$

advective:

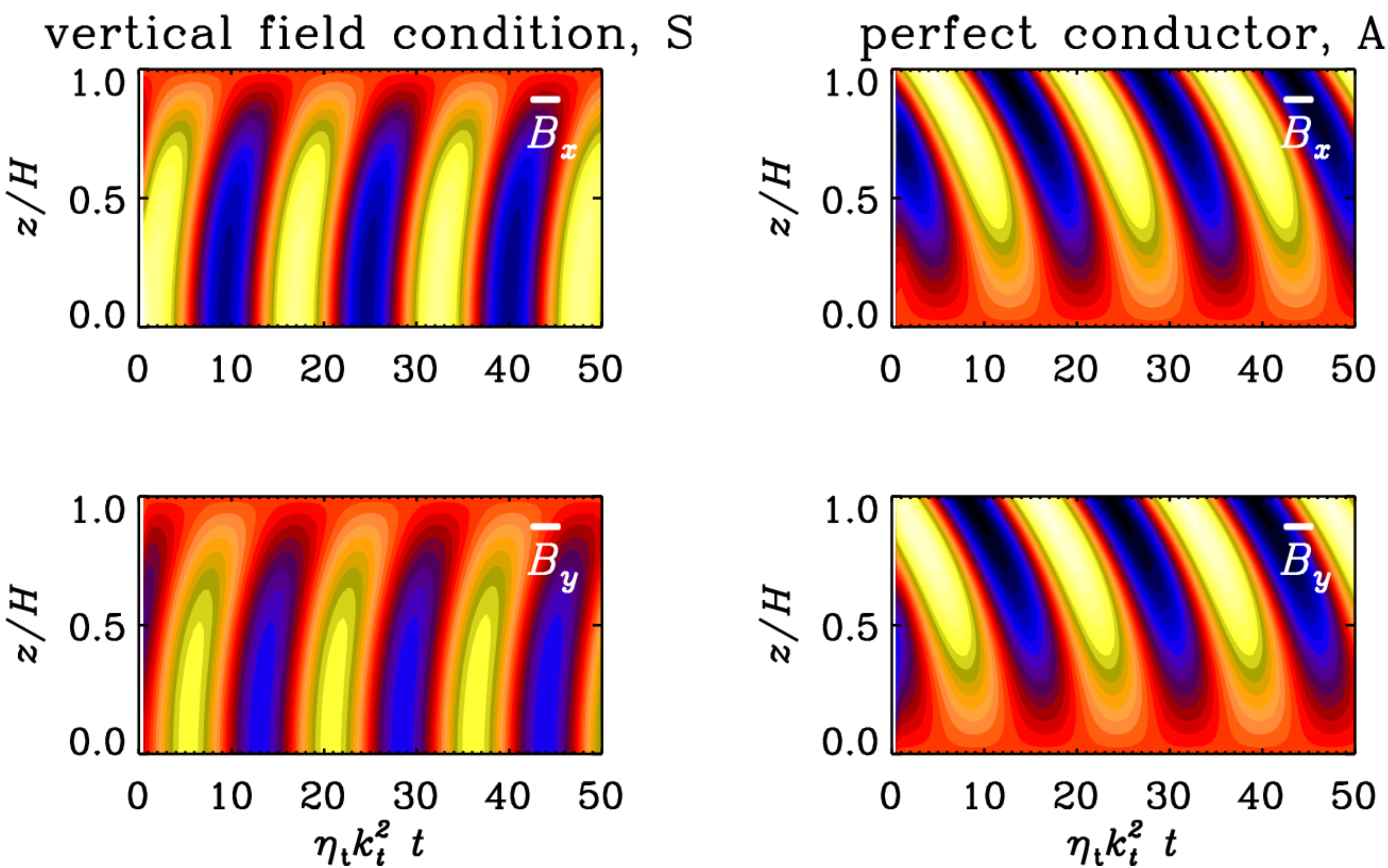
$$\alpha_{\rm M} \overline{\boldsymbol{U}}$$

Helical forcing profile:



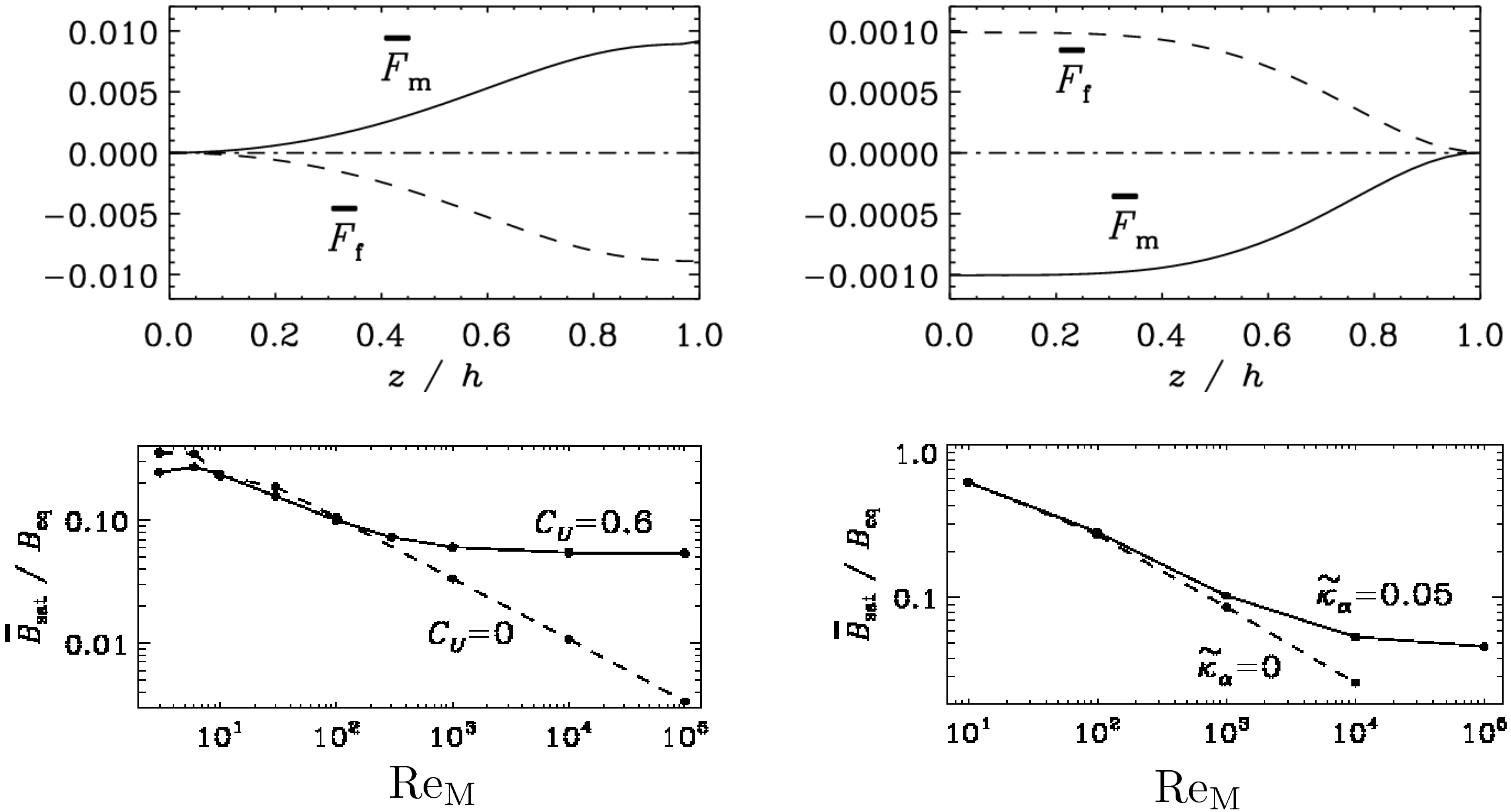


Dynamo Waves

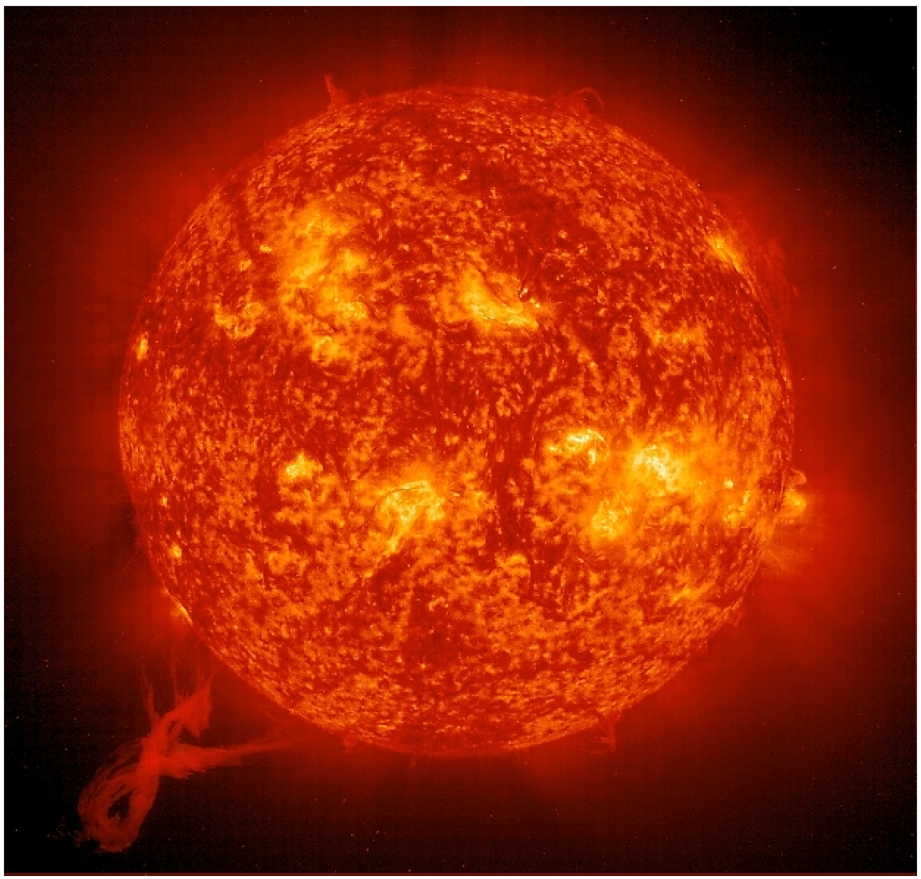


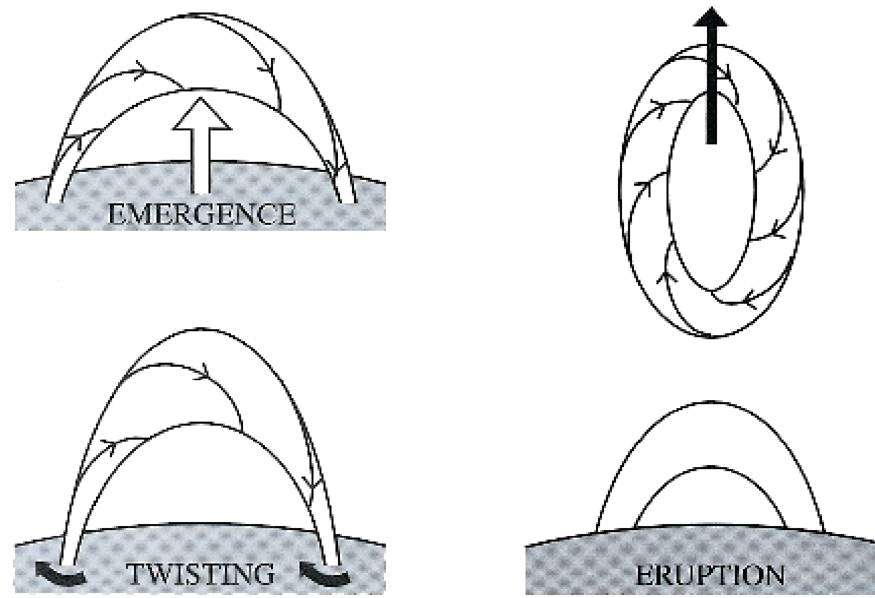
Magnetic Felicity Fluxes

open boundary symmetric wind	vs.	closed boundary antisymmetric κ_α



Twisted Magnetic Fields





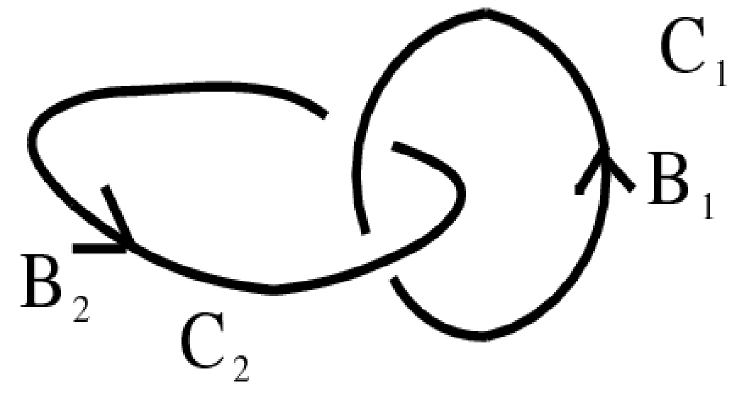
Twisted fields are more likely to erupt *(Canfield et al. 1999)*.

Magnetic Helicity

Measure for the topology:

$$H_{\mathrm{M}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 2n\phi_1\phi_2$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B} \qquad \phi_i = \int_{S_i} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$



$n =$ number of mutual linking

Conservation of magnetic helicity:

$$\lim_{\eta \to 0} \frac{\partial}{\partial t} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = 0 \qquad \eta = \text{magnetic resistivity}$$

Realizability condition:

$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

→ Magnetic energy is bound from below by magnetic helicity.

Equilibrium States

Ideal MHD: $\eta = 0$

Induction equation: $\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B})$

Task: Find the state with minimal energy.
Constraint: magnetic helicity conservation

	constraint	equilibrium
Woltjer (1958):	$\frac{\partial}{\partial t} \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha \boldsymbol{B}$
Taylor (1974):	$\frac{\partial}{\partial t} \int_{\tilde{V}} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha(a, b) \boldsymbol{B}$

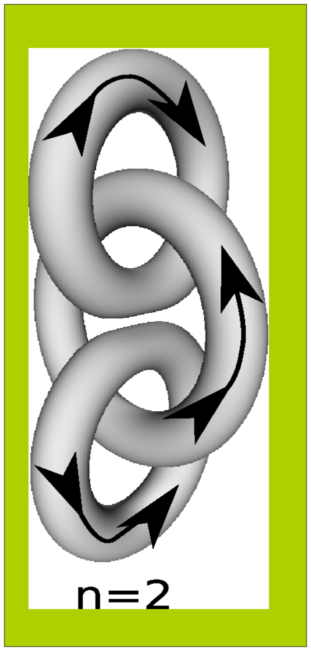
constant along field line

V total volume $\tilde{V}$ volume along magnetic field line

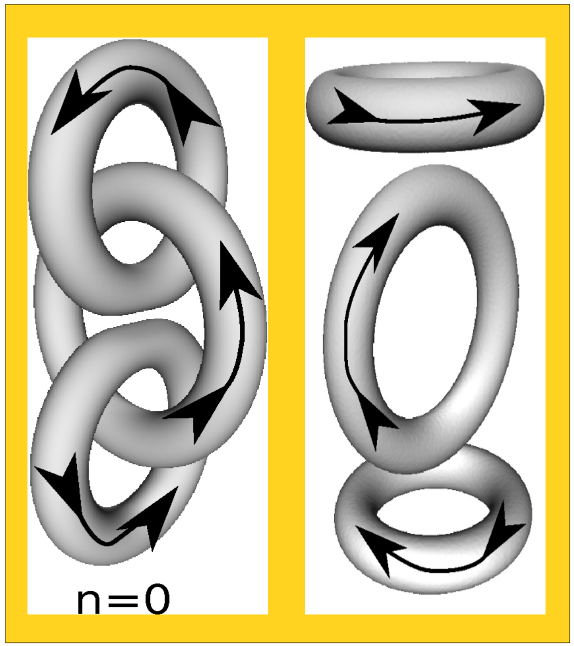
Interlocked Flux Rings

actual linking vs. magnetic helicity

$H_{\rm M} \neq 0$



$H_{\rm M} = 0$



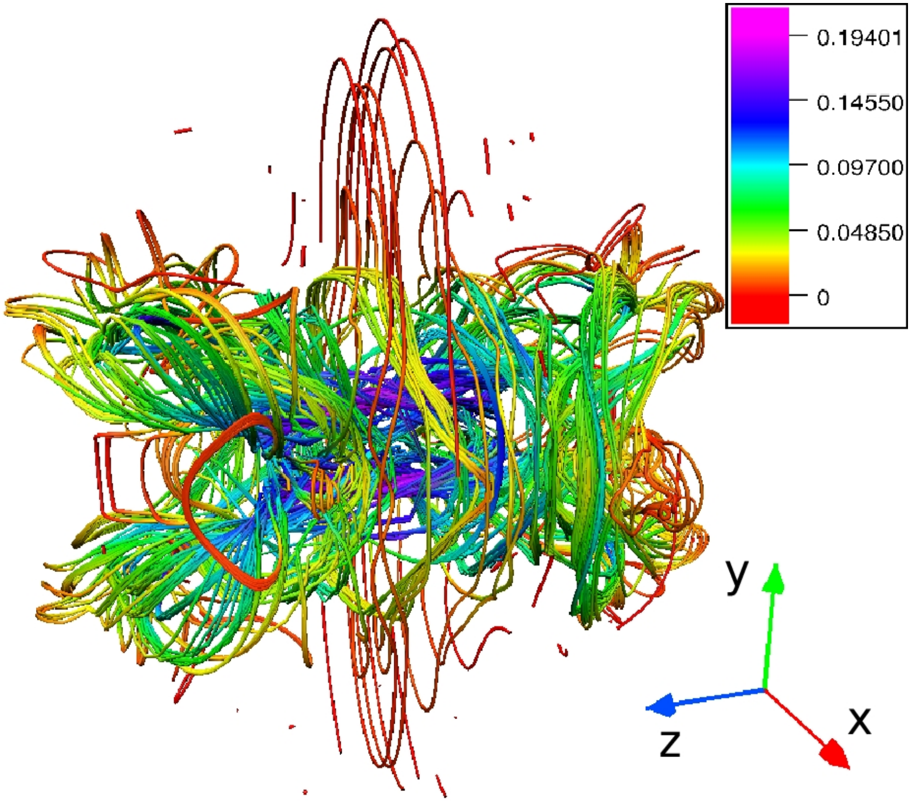
- initial condition: flux tubes
- isothermal compressible gas
- viscous medium
- periodic boundaries

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{U} \times \boldsymbol{B} + \eta \nabla^2 \boldsymbol{A} \qquad \frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \boldsymbol{U}$$

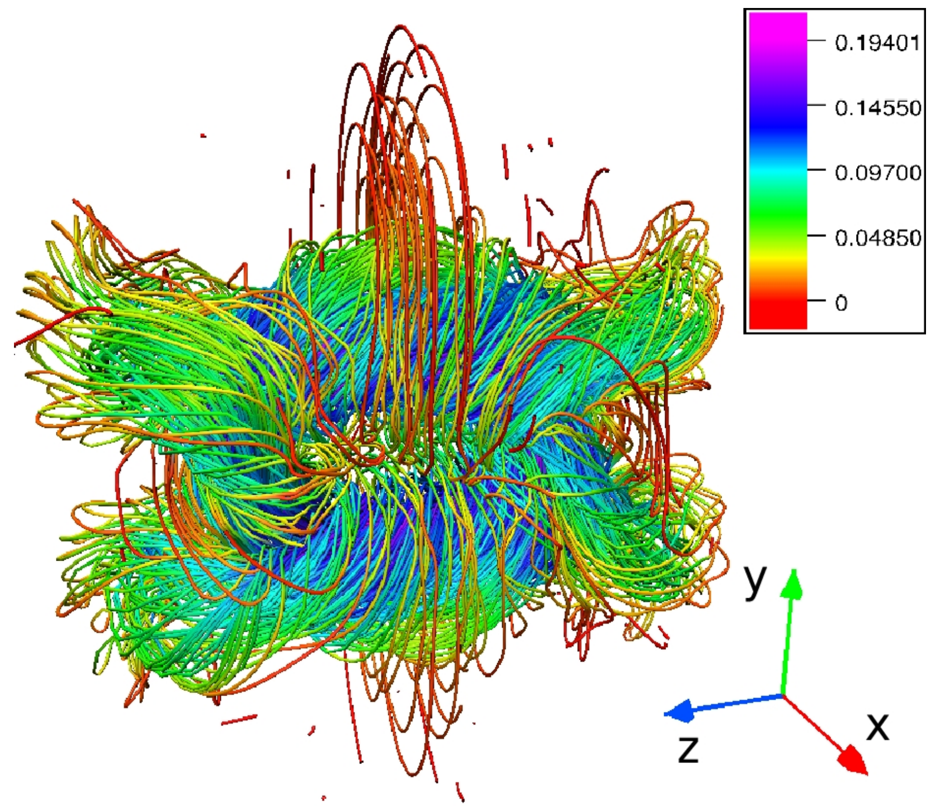
$$\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t} = -c_{\rm S}^2 \boldsymbol{\nabla} \ln \rho + \boldsymbol{J} \times \boldsymbol{B}/\rho + \boldsymbol{F}_{\rm visc}$$

Interlocked Flux Rings

$\tau = 4$

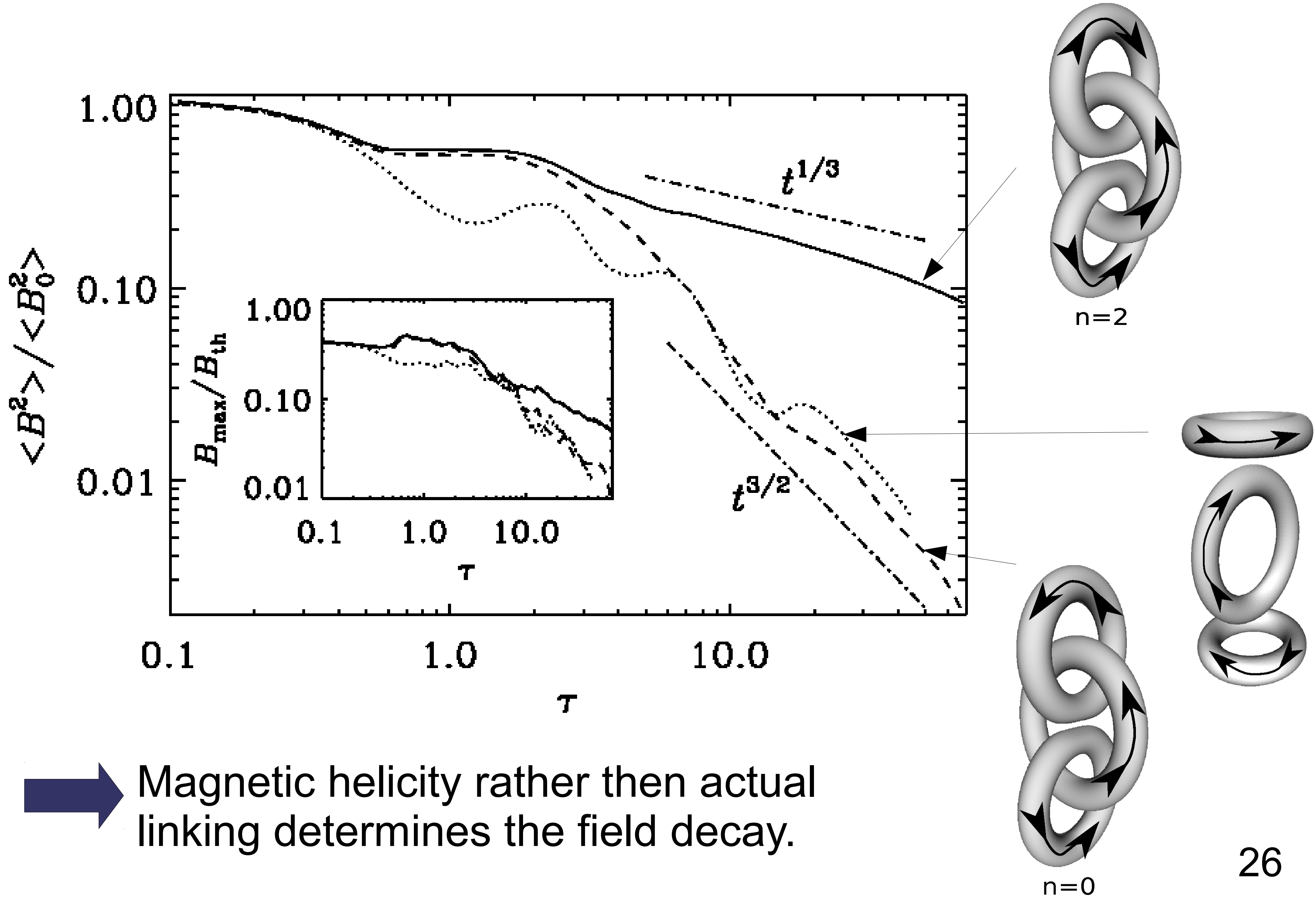


$H_{\mathrm{M}} = 0$



$H_{\mathrm{M}} \neq 0$

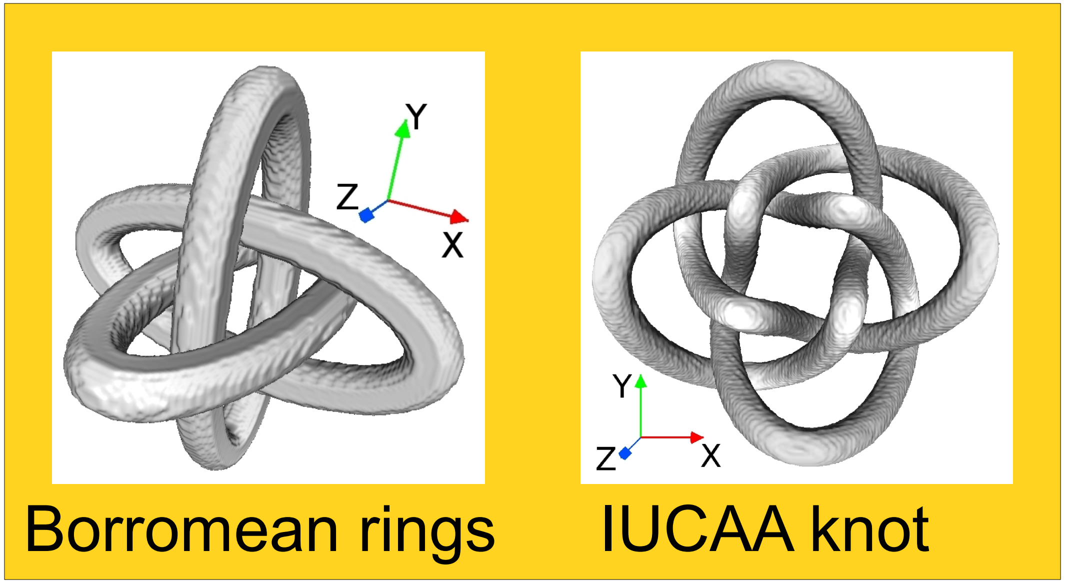
Interlocked Flux Rings



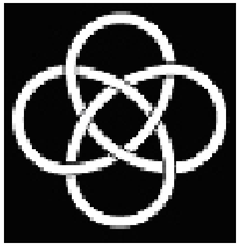
➡ Magnetic helicity rather then actual linking determines the field decay.

IUCAA Knot and Borromean Rings

- Is magnetic helicity sufficient?
- Higher order invariants?

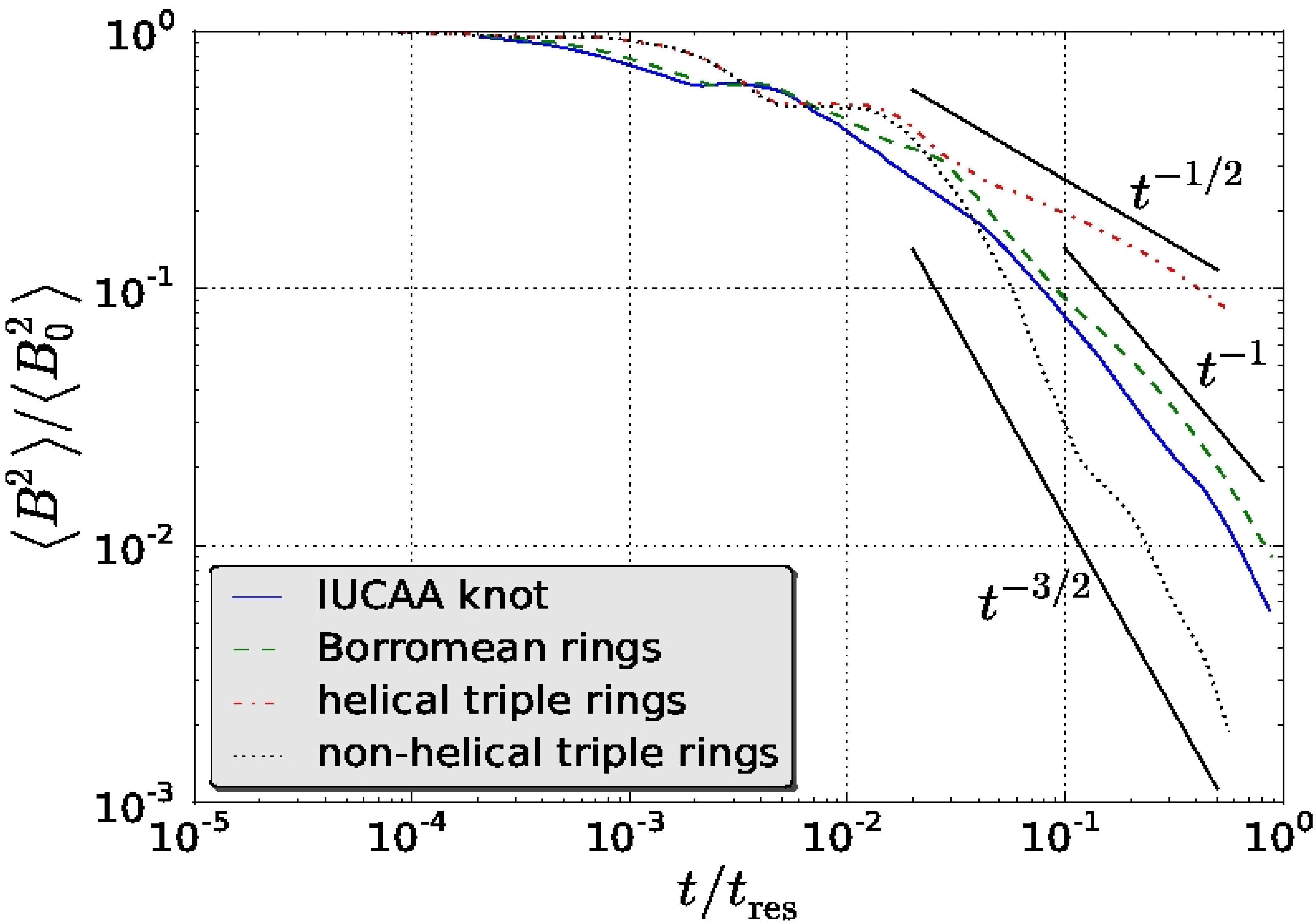


$$H_{\mathrm{M}} = 0$$



IUCAA = The Inter-University Centre for Astronomy and Astrophysics, Pune, India

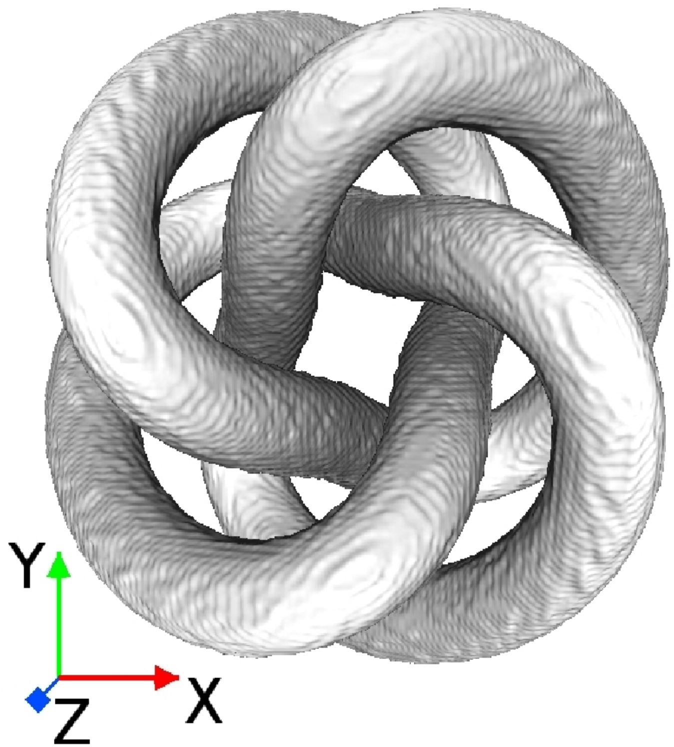
Magnetic Energy Decay



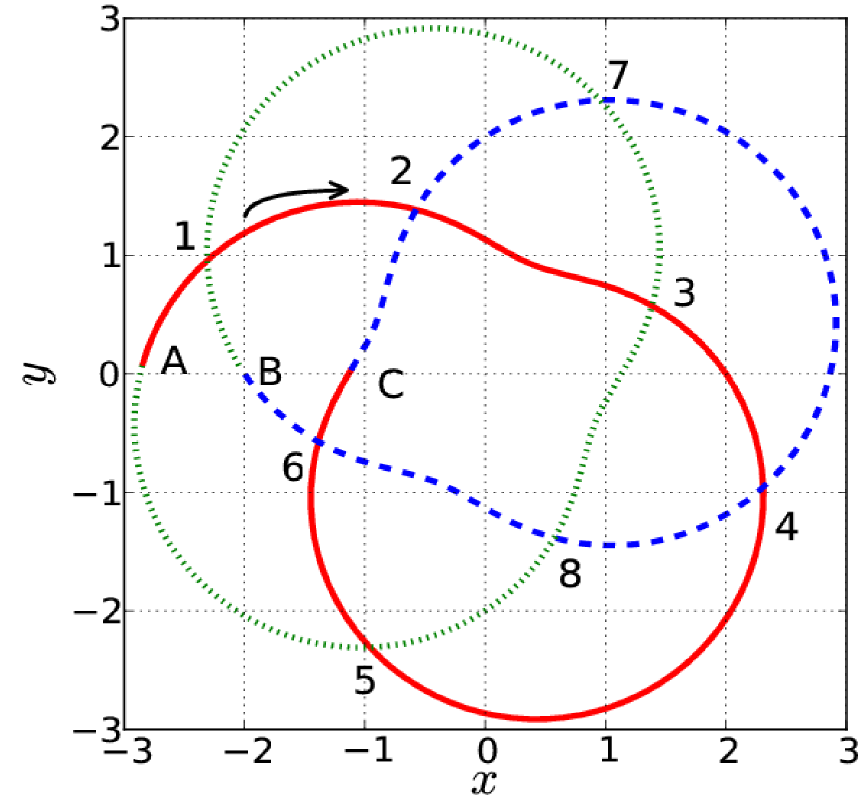
Higher order invariants?

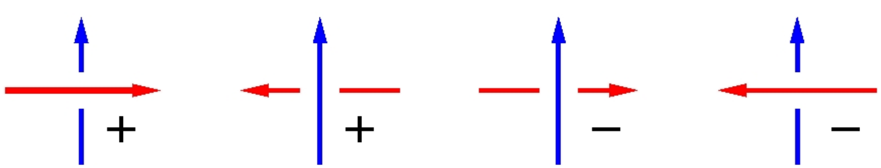
Braid Representation

need $B_z > 0$ → braid representation of knots and links

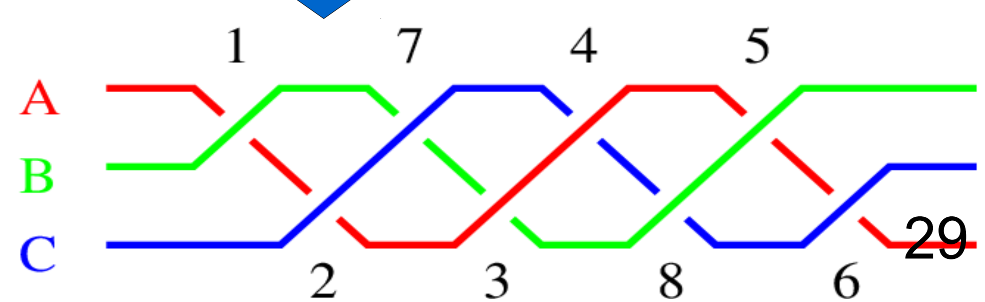


4-foil knot





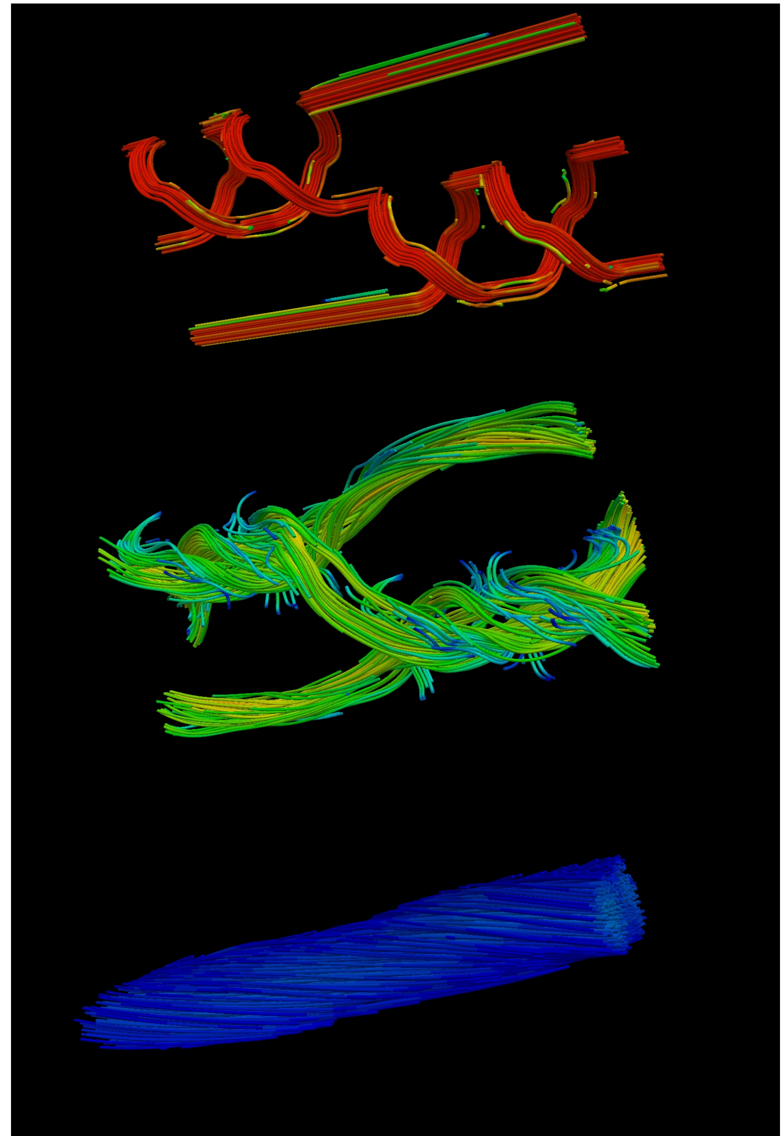
$$n_{\text{linking}} = (n_+ - n_-)/2$$



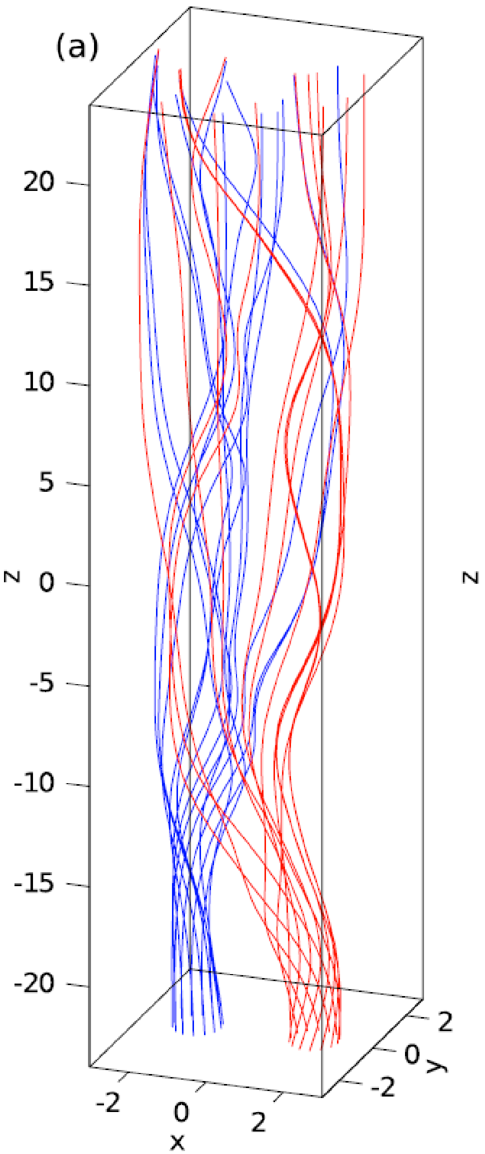
Magnetic Braid Configurations

AAA (trefoil knot)

AABB (Borromean rings)



Fixed Point Index



Yeates et al. 2011

Trace magnetic field lines from z_0 to z.

mapping: $(x, y) \rightarrow \mathbf{F}_z(x, y)$

Color coding:

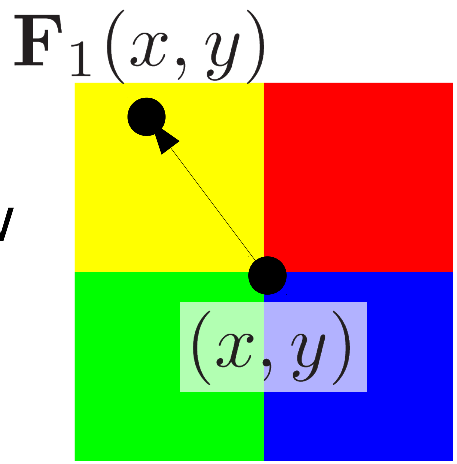
Compare (x, y) with $\mathbf{F}_1(x, y)$:

$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y > y$ red

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y > y$ yellow

$\mathbf{F}_1^x < x, \quad \mathbf{F}_1^y < y$ green

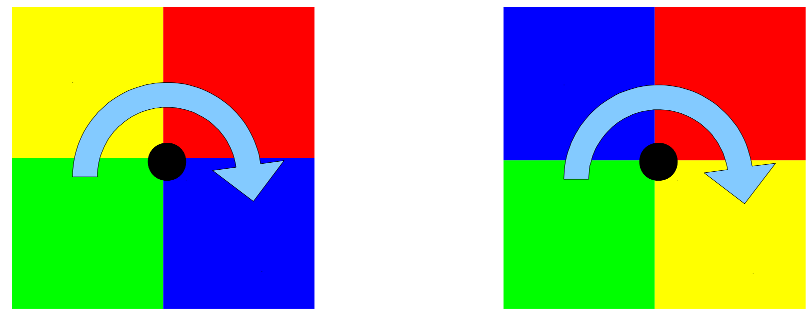
$\mathbf{F}_1^x > x, \quad \mathbf{F}_1^y < y$ blue



Fixed Point Index

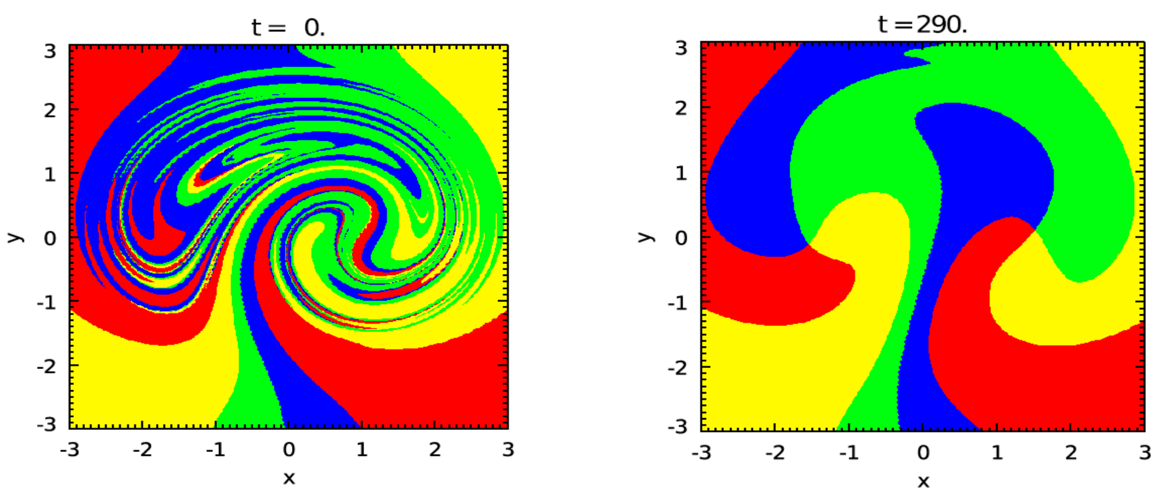
fixed points: $\mathbf{F}_1(x,y) = (x,y)$

Sign t_i of fixed point i :



$t_i = +1$ $\qquad$ $t_i = -1$

Fixed point index: $T = \sum_i t_i$ conserved for $\lim \eta \to 0$



Taylor state is not reached
$\rightarrow$ T is additional constraint

Summary

- Helical turbulence can drive large-scale dynamo action.
- Convective motions in plasma drive dynamos.
- Dynamical alpha-quenching as more self-consistent model.

- Advective magnetic helicity fluxes can alleviate catastrophic quenching.
- Diffusive magnetic helicity fluxes can alleviate catastrophic quenching.

- Braiding increases stability through the *realizability condition*.
- Fixed point index as additional constraint in relaxation.

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Appendix

Viscous force: $\boldsymbol{F}_{\rm visc} = \rho^{-1}\boldsymbol{\nabla}\cdot 2\nu\rho\mathbf{S}$

Strain tensor: $\mathsf{S}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{U}$

Sound speed: $c_{\rm S} = \sqrt{\gamma\frac{p}{\rho}}$