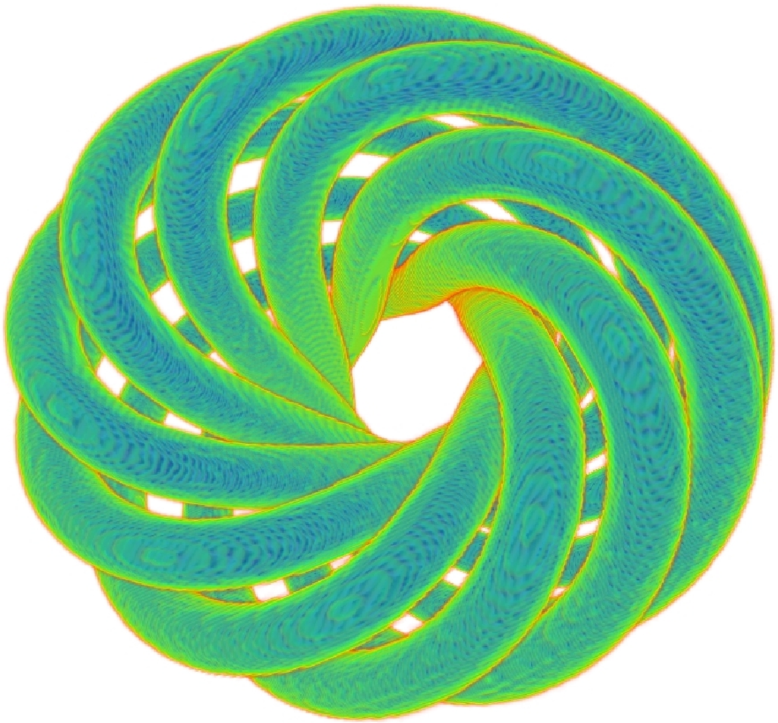
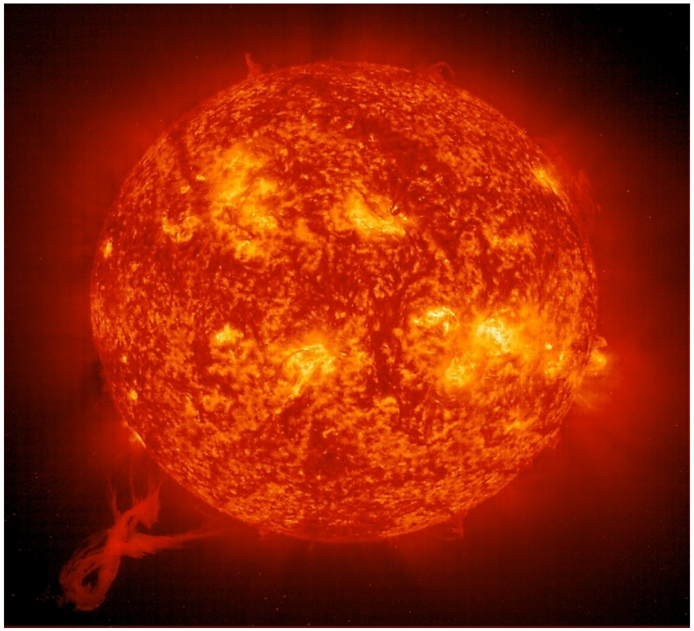
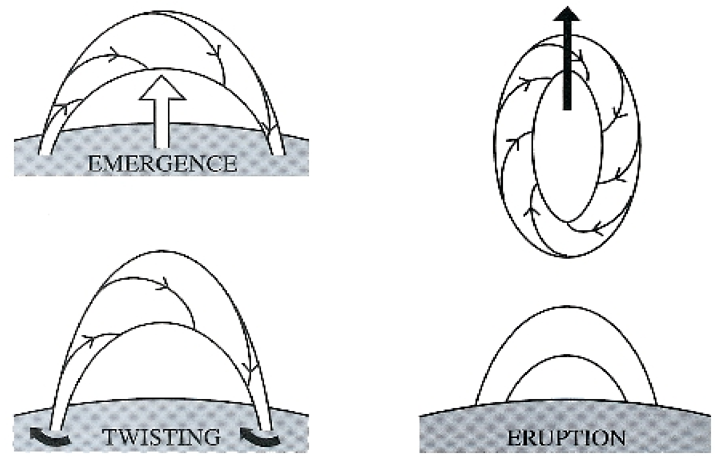
Topological constraints in magnetic field relaxation

Simon Candelaresi

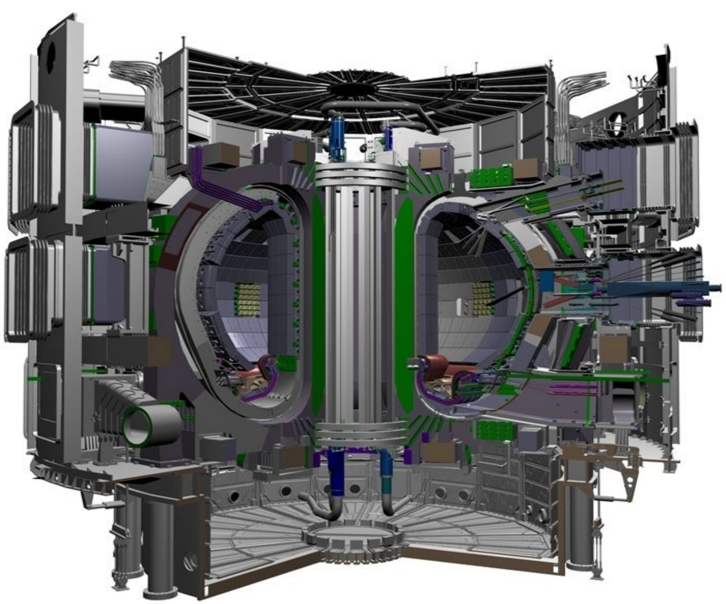


Twisted Magnetic Fields





- Twisted fields are more likely to erupt *(Canfield et al. 1999)*.



- Twist increases the stability of magnetic fields in tokamaks.

Twisted Field in the Sun

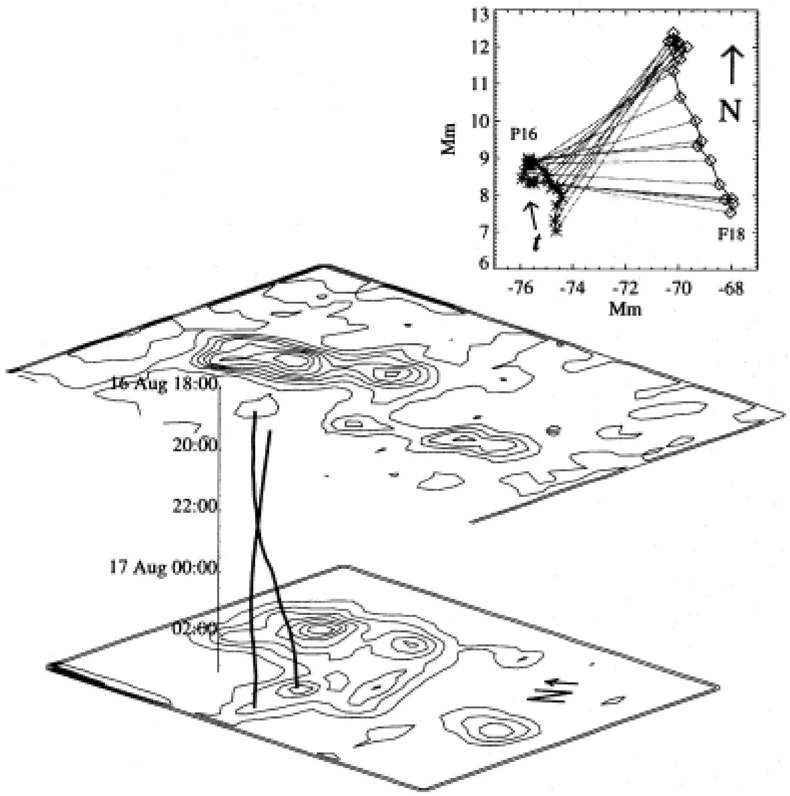
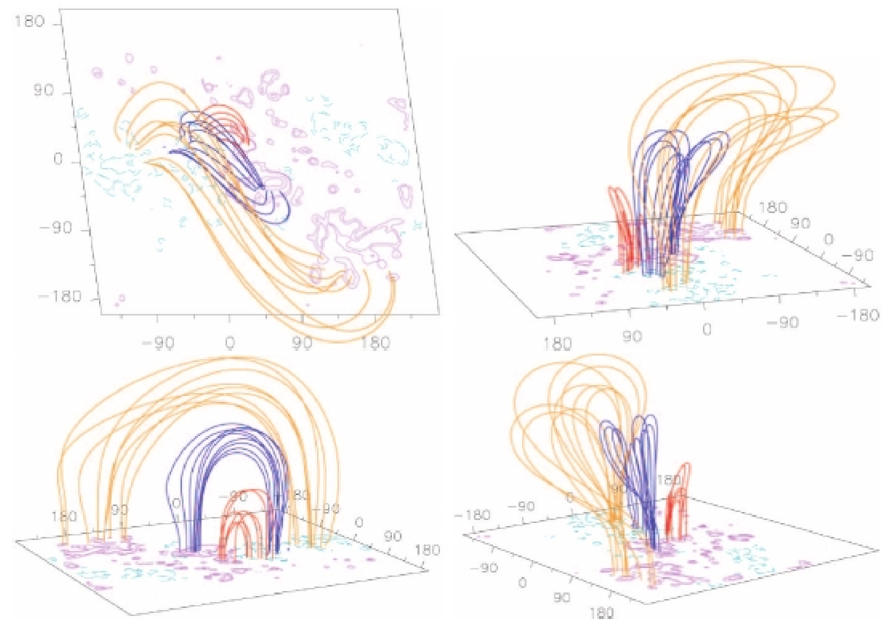


FIG. 3b

Magnetic bipoles' movement on the Sun's surface.
(Leka et al. 1996)



Force-free extrapolation of the photospheric magnetic field from 1999, August 21.
(Gibson et al. 2002)

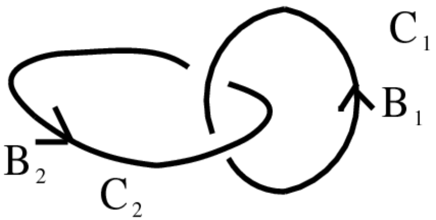
Force free condition:

$$\nabla \times B = \alpha B$$
$$J \times B = 0$$

Magnetic Helicity

$$H_{\mathrm{M}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 2n\phi_1\phi_2$$

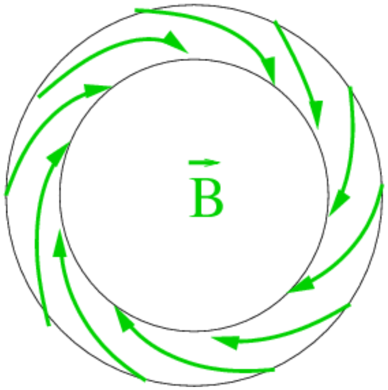
$$\phi_i = \int_{S_i} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$



Realizability condition:

$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

Magnetic energy is bound from below by magnetic helicity.



twisted field

magnetic helicity conservation

$$\mathrm{Re}_{\mathrm{M}} \to \infty$$

$$\frac{\mathrm{d}H_{\mathrm{M}}}{\mathrm{d}t} = 0$$

trefoil knot

Stability Criteria

Ideal MHD: $\eta = 0$

↓

Induction equation: $$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B})$$

	constraint	equilibrium
Woltjer (1958):	$\frac{\partial}{\partial t} \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha \boldsymbol{B}$
Taylor (1974):	$\frac{\partial}{\partial t} \int_{\tilde{V}} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha(a,b) \boldsymbol{B}$

$\alpha(a,b)$: constant along field line

V total volume $\tilde{V}$ volume along magnetic field line

Creation of Magnetic Field and Magnetic Helicity

Mean-field decomposition: $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$

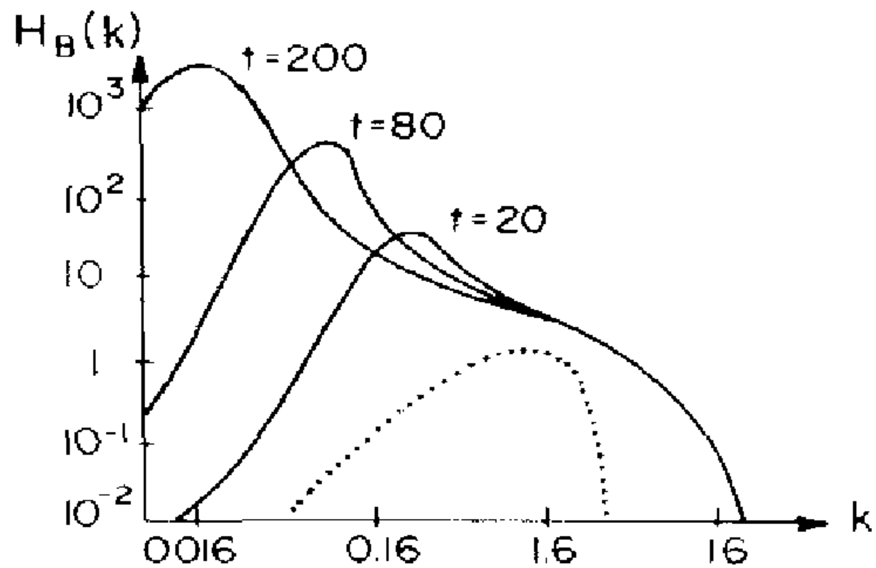
Induction equation: $\partial_t \overline{\boldsymbol{B}} = \eta \nabla^2 \overline{\boldsymbol{B}} + \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{\mathcal{E}}})$

Electromotive force: $\overline{\boldsymbol{\mathcal{E}}} \underset{\alpha}{=} \overline{\boldsymbol{u} \times \boldsymbol{b}} = \alpha \overline{\boldsymbol{B}} - \eta_{\mathrm{t}} \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}$

α effect: $\alpha = \alpha_{\mathrm{K}} + \alpha_{\mathrm{M}} = -\tau \overline{\boldsymbol{\omega} \cdot \boldsymbol{u}}/3 + \overline{\boldsymbol{j} \cdot \boldsymbol{b}}/(3\overline{\rho})$

Inverse cascade:

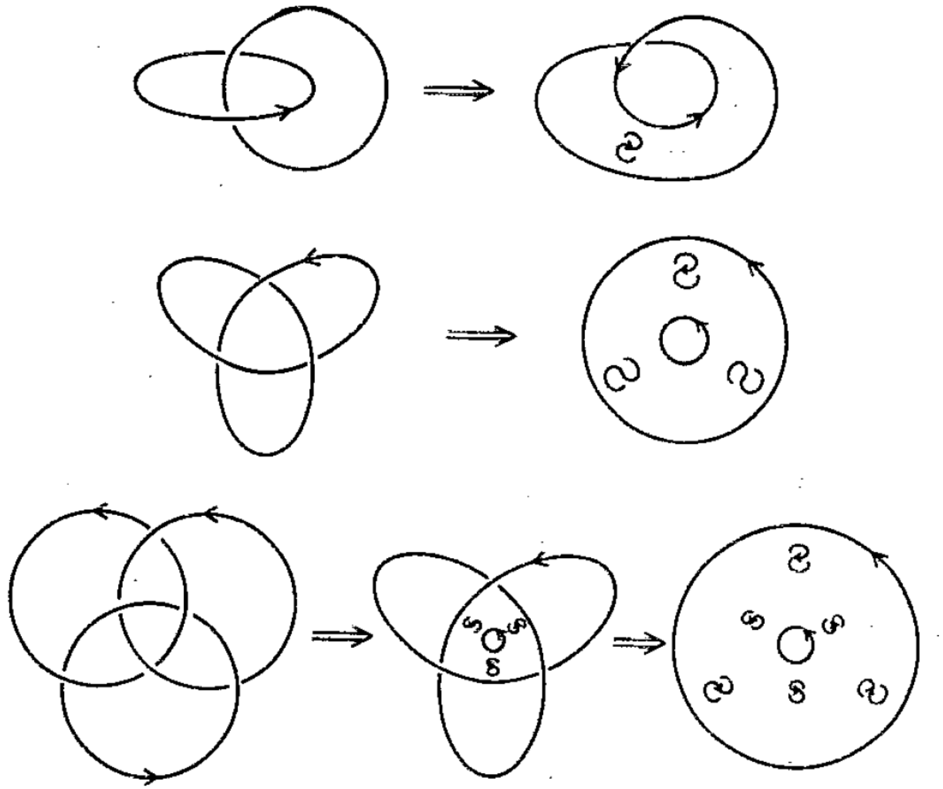
Large- and small-scale magnetic helicity of opposite sign is created.
(Pouquet et al. 1976)



Leorat et al. 1975

Reconnection Characteristics

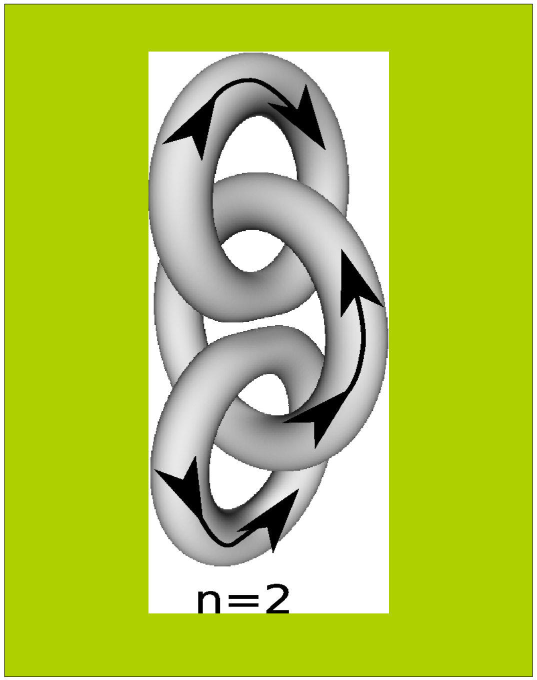
Conversion of linking into twisting:



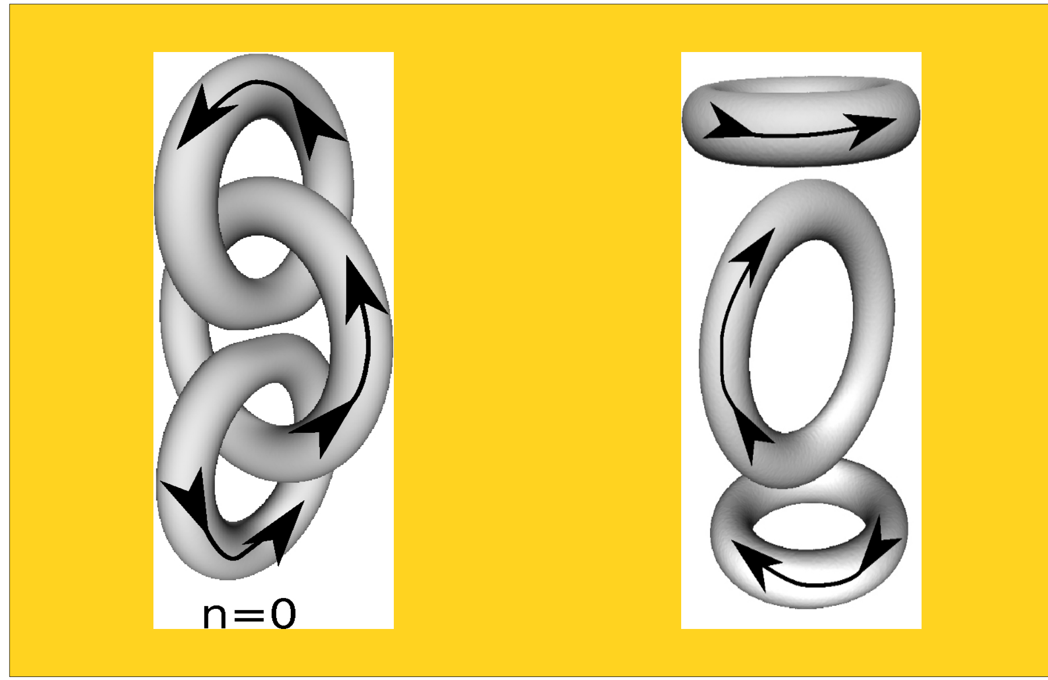
Ruzmaikin and Akhmetiev, 1994

Interlocked Flux Rings

$H_{\mathrm{M}} \neq 0$



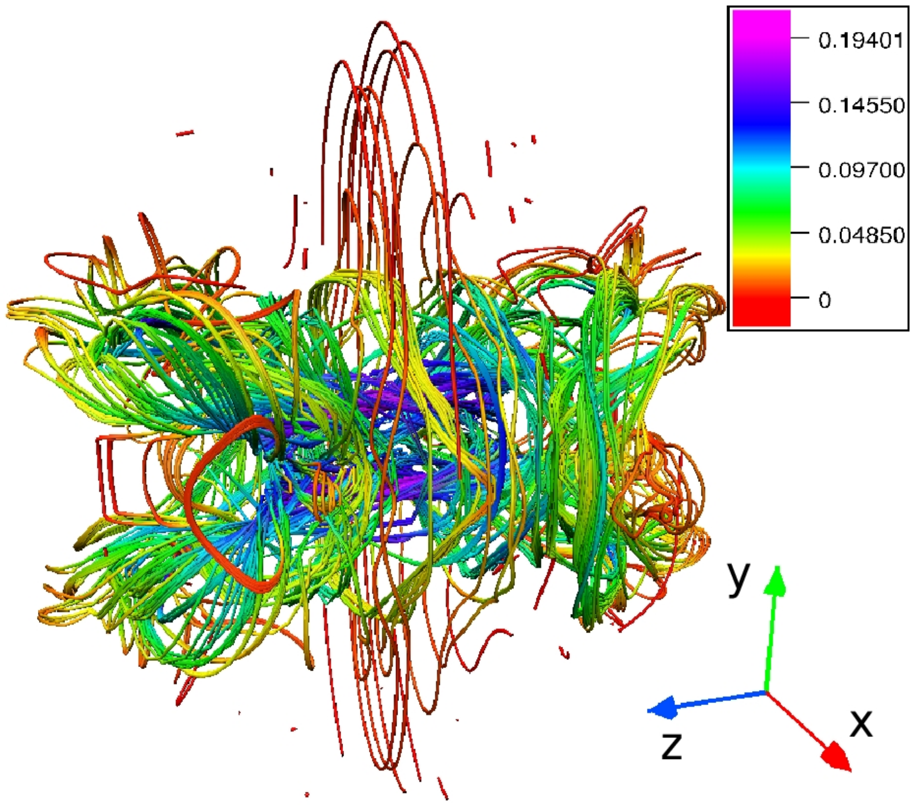
$H_{\mathrm{M}} = 0$



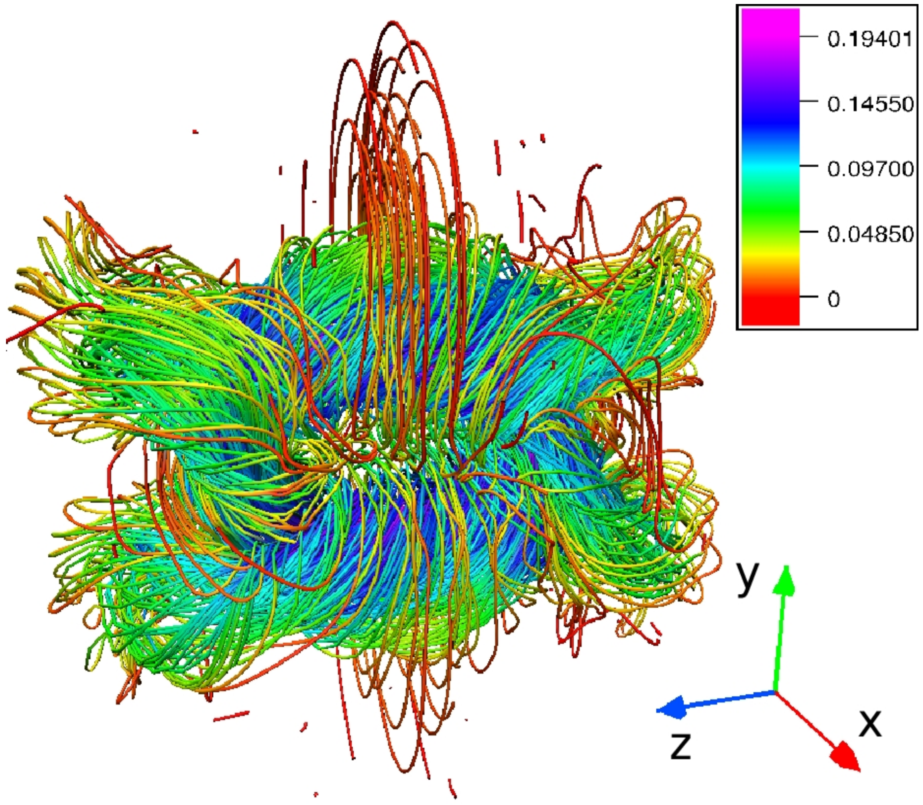
- isothermal compressible gas
- viscous medium
- periodic boundaries

Interlocked Flux Rings

$\tau = 4$

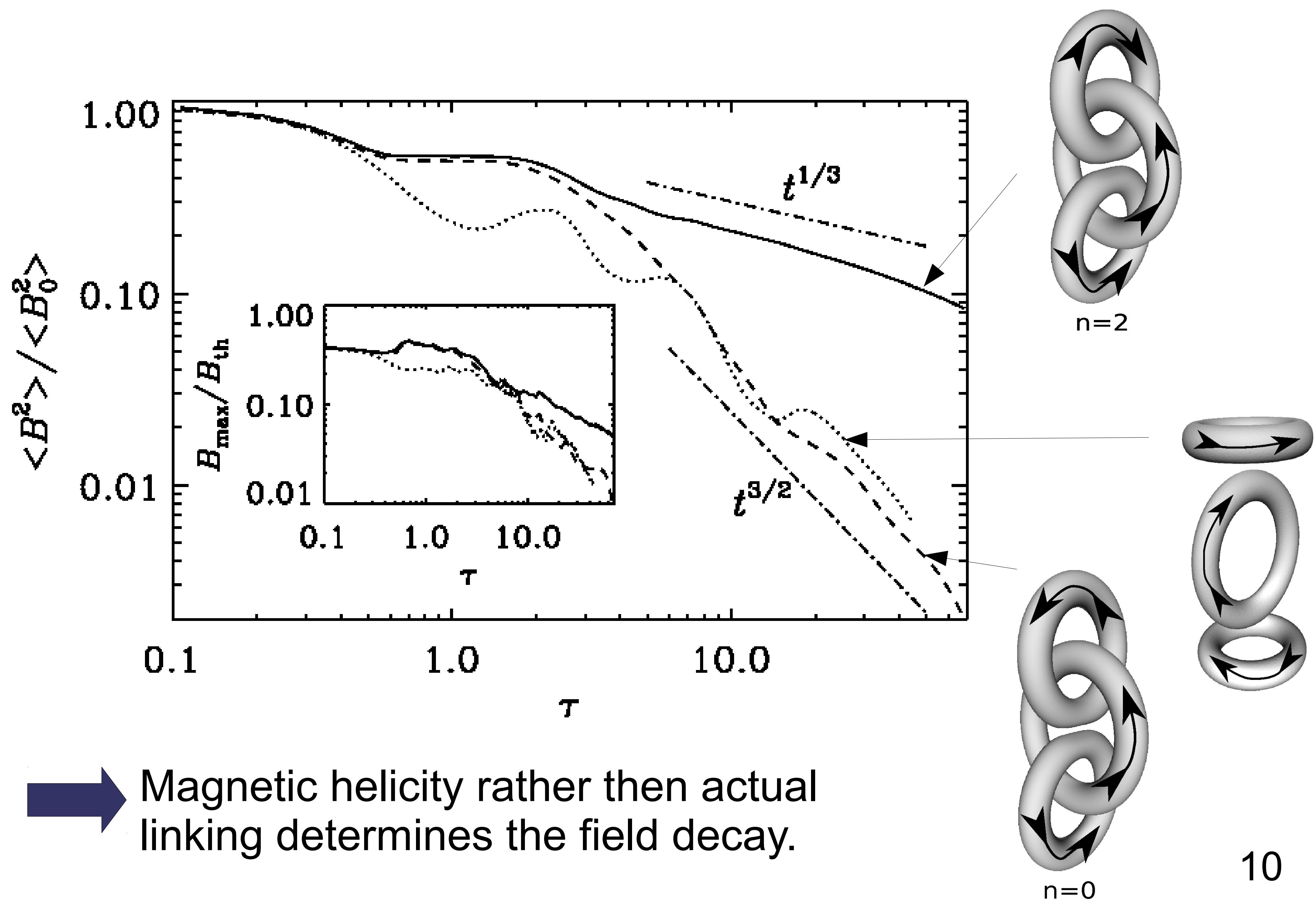


$H_\mathrm{M} = 0$



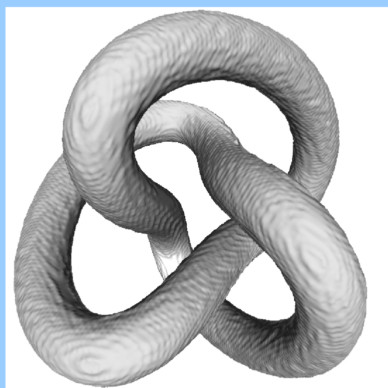
$H_\mathrm{M} \neq 0$

Interlocked Flux Rings

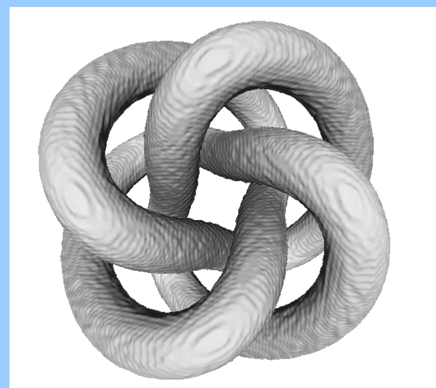


Magnetic helicity rather then actual linking determines the field decay.

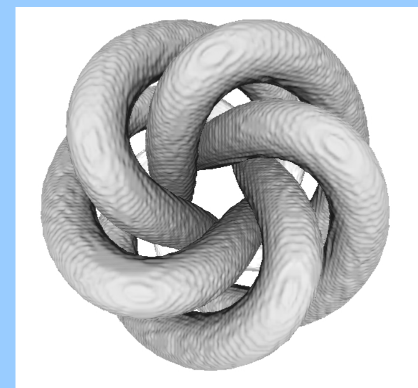
N-foil Knots



3-foil



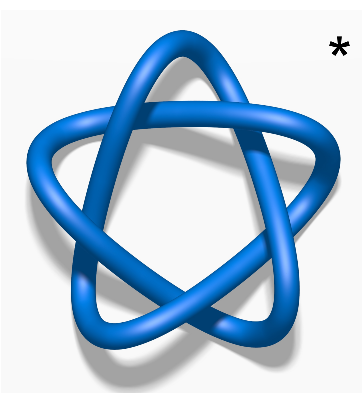
4-foil



5-foil

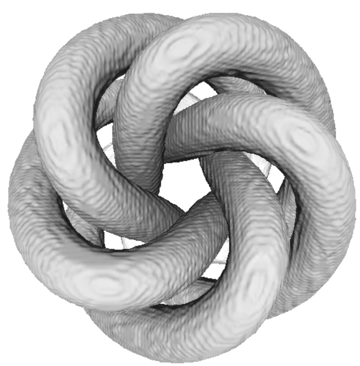
6-foil

7-foil



cinquefoil knot

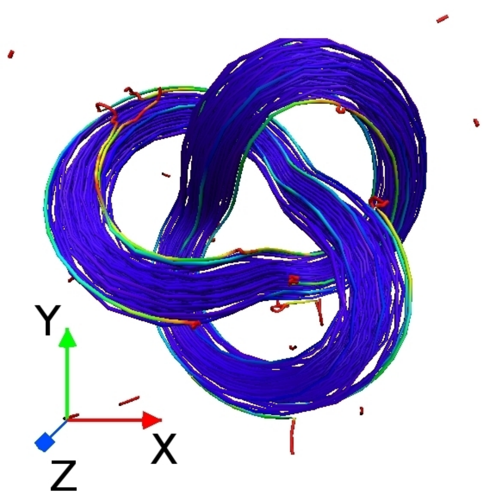
$\neq$



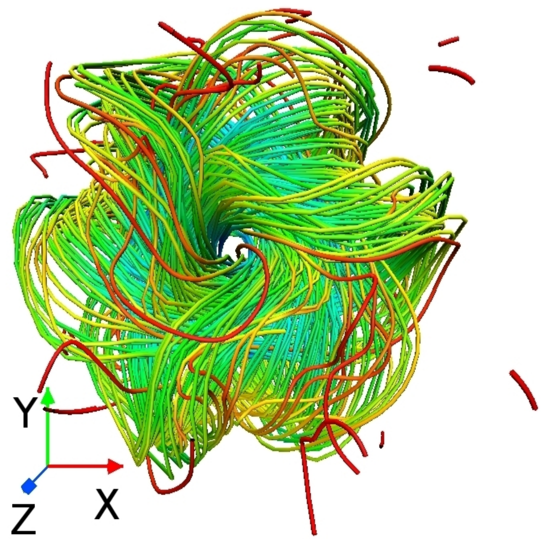
$$x(s) = \begin{pmatrix} (C + \sin s n_{\mathrm{f}}) \sin[s(n_{\mathrm{f}} - 1)] \\ (C + \sin s n_{\mathrm{f}}) \cos[s(n_{\mathrm{f}} - 1)] \\ D \cos s n_{\mathrm{f}} \end{pmatrix}$$

* from Wikipedia, author: Jim.belk

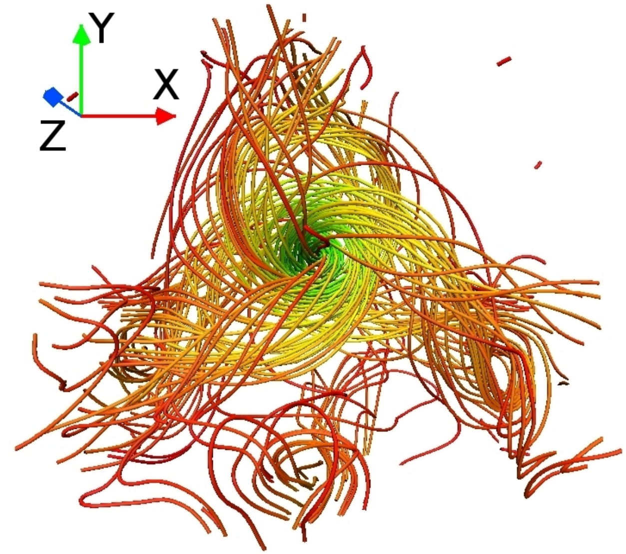
N-foil Knots



$t = 0$



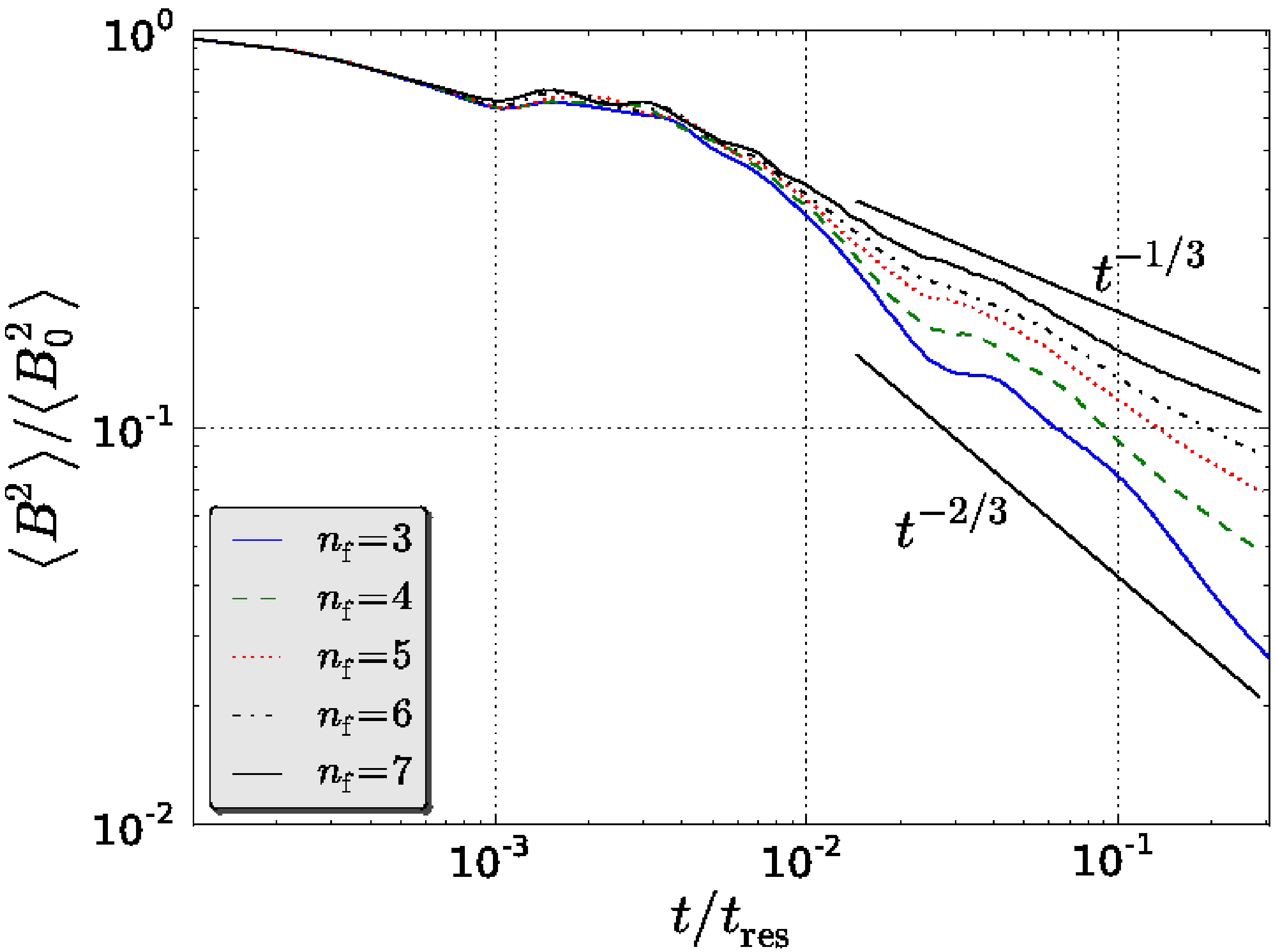
$t = 6$



$t = 39$

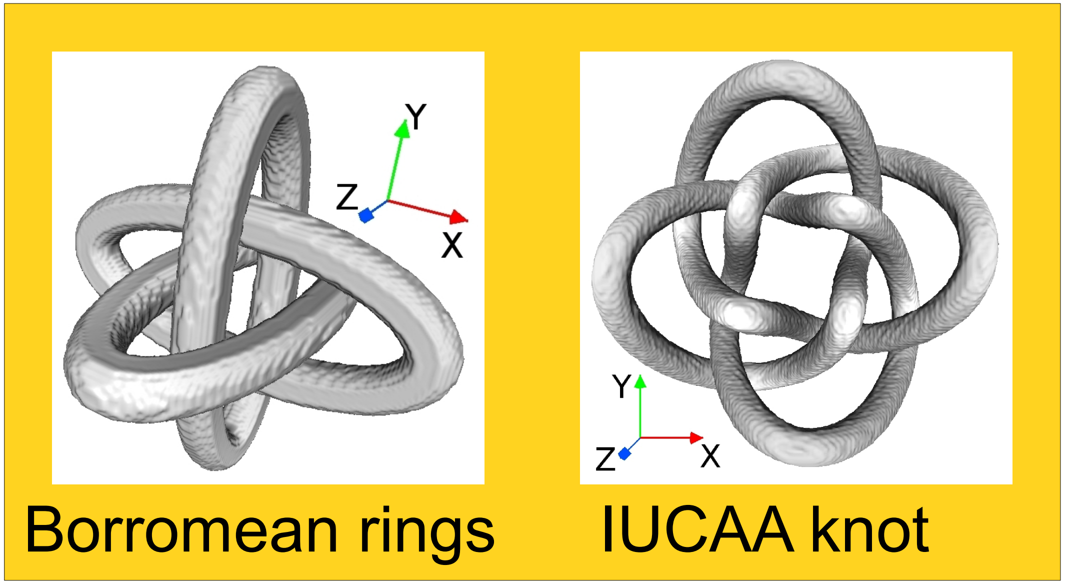
- Magnetic helicity is approximately conserved.
- Self-linking is transformed into twisting after reconnection.

N-foil Knots



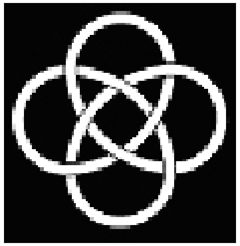
Slower decay for higher $n_{\rm f}$.

IUCAA Knot and Borromean Rings



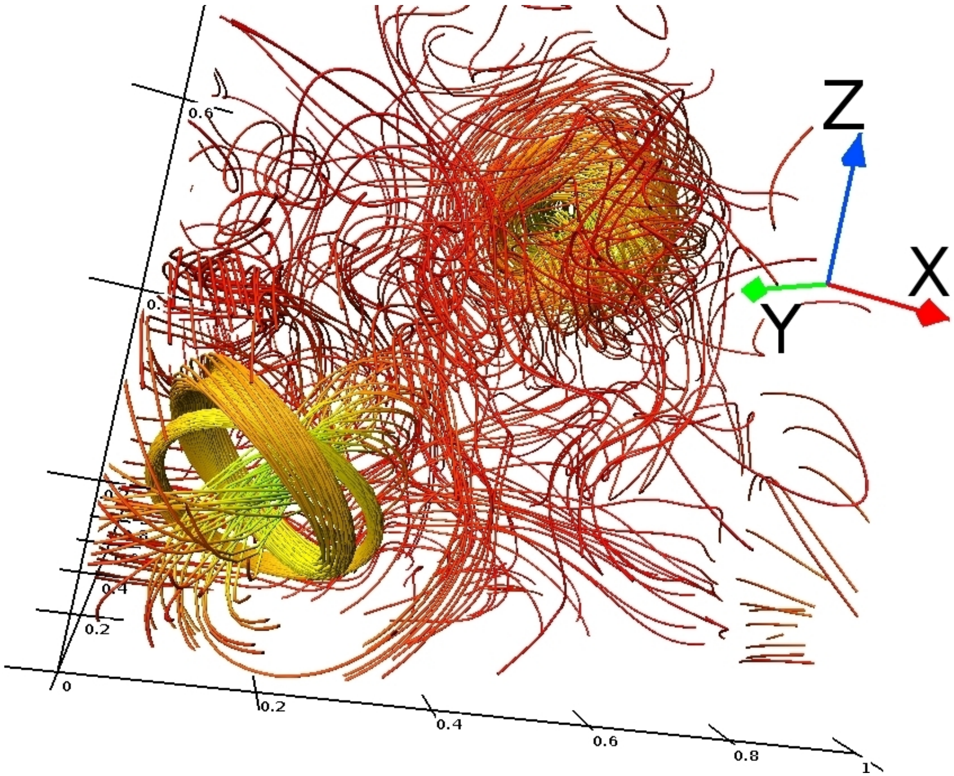
$$H_{\mathrm{M}} = 0$$

- Is magnetic helicity sufficient?
- Higher order invariants?

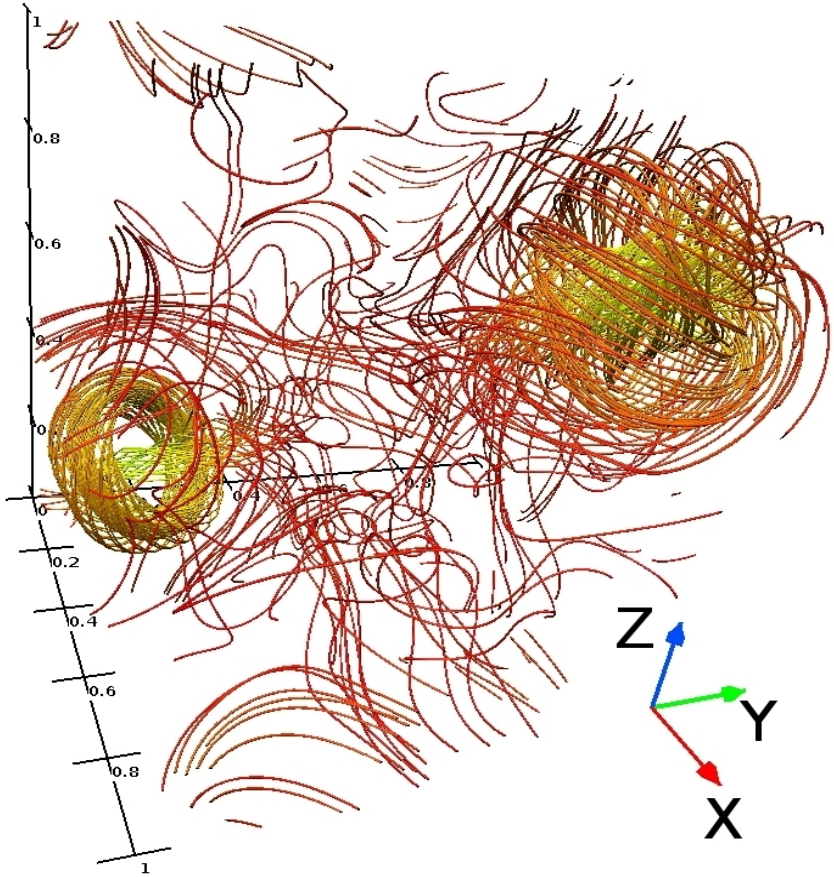


IUCAA = The Inter-University Centre for Astronomy and Astrophysics, Pune, India

Reconnection Characteristics



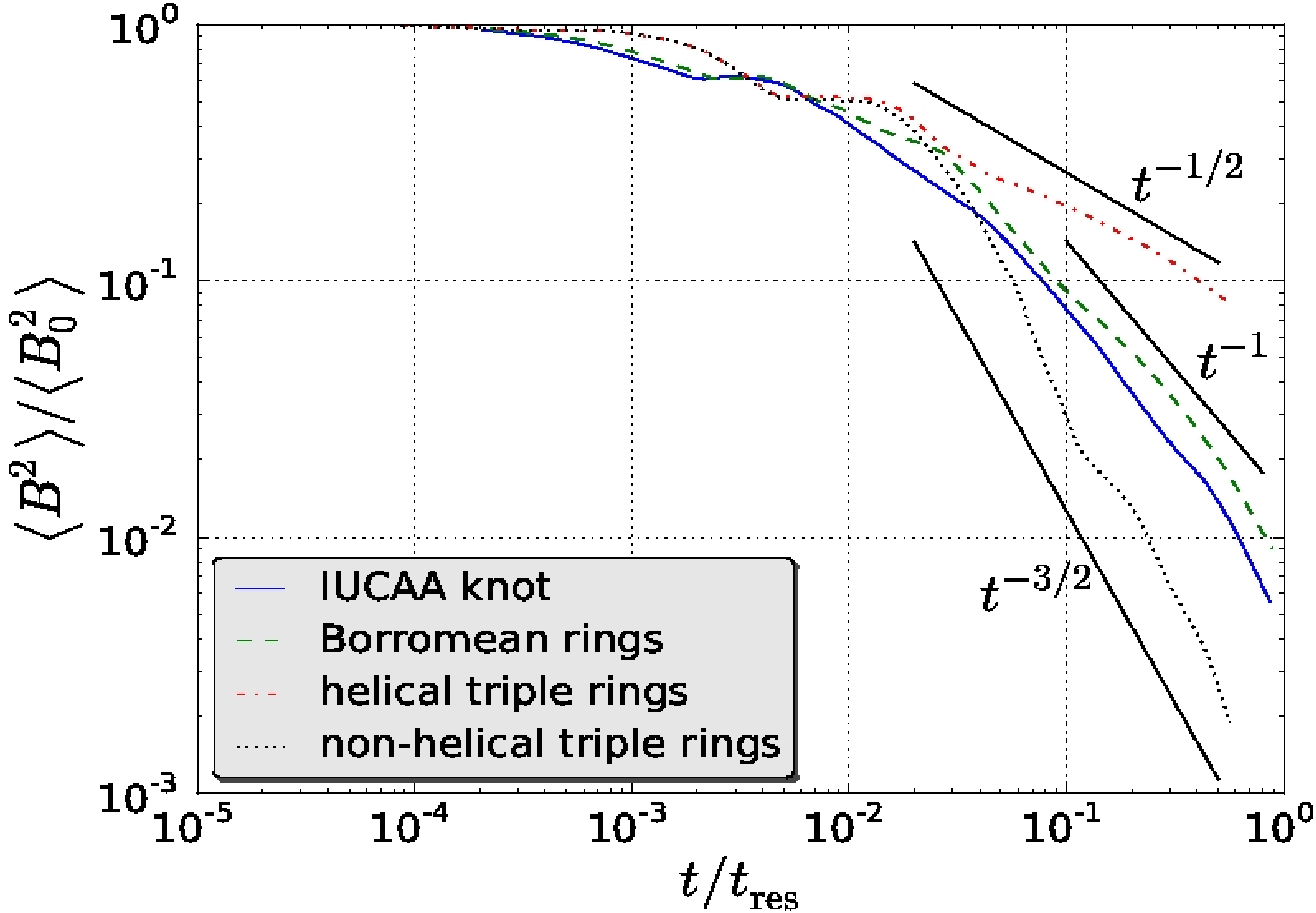
$t = 70$



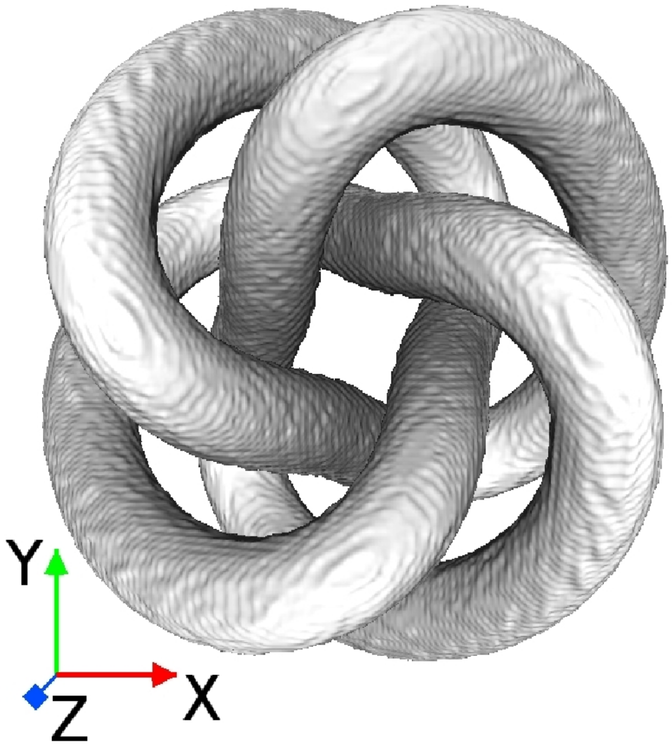
$t = 78$

3 rings → Twisted ring + interlocked rings → 2 twisted rings

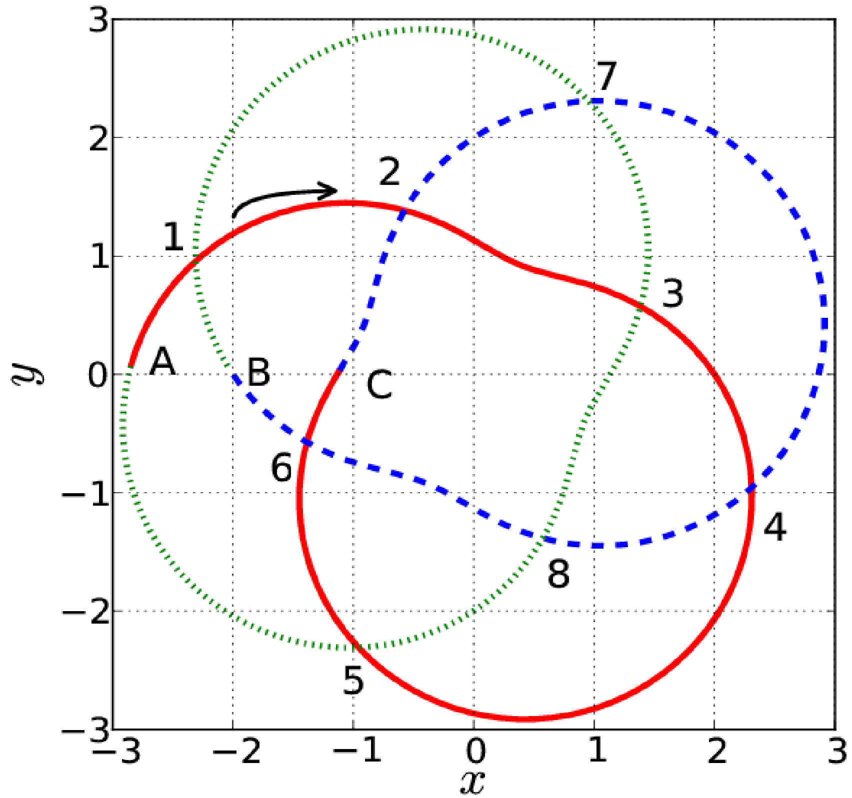
Magnetic Energy Decay

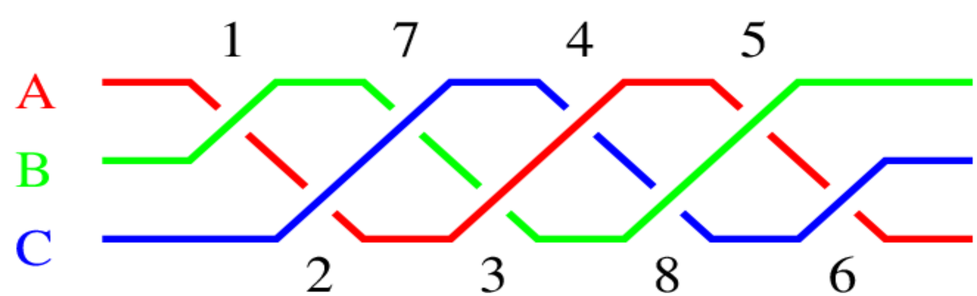


Braid Representation



4-foil knot



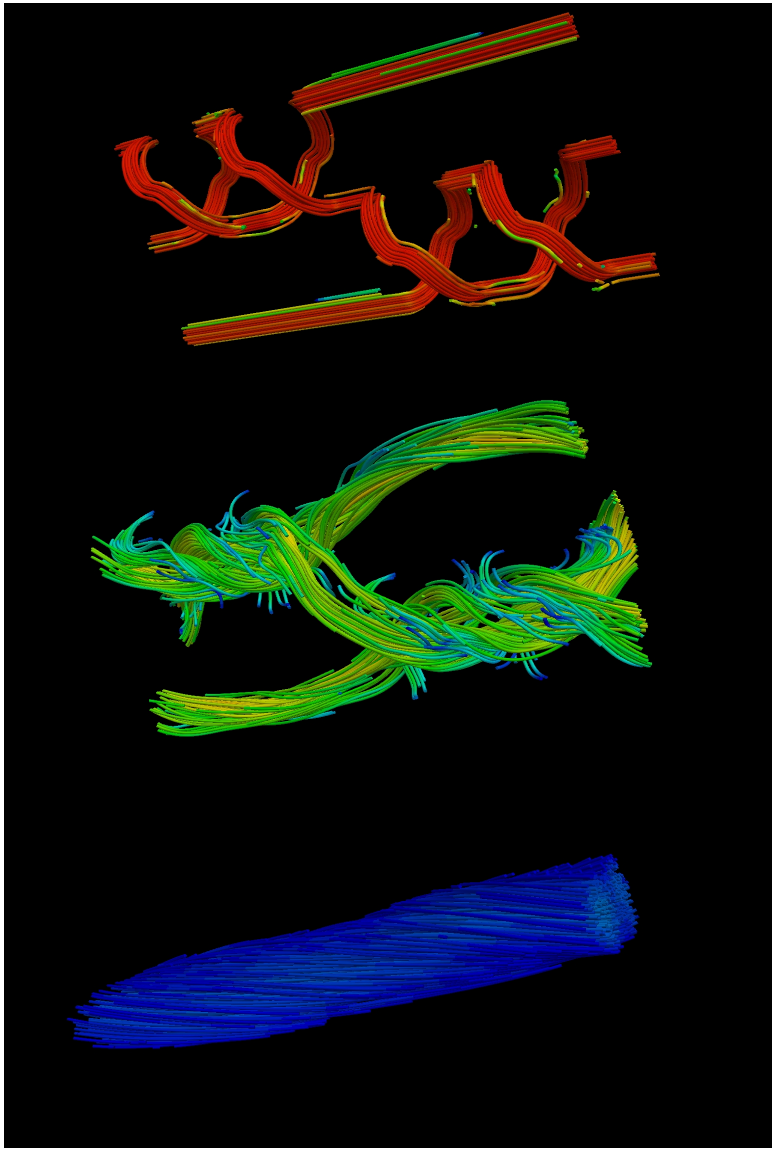


Word: ABABABAB $B_z > 0$

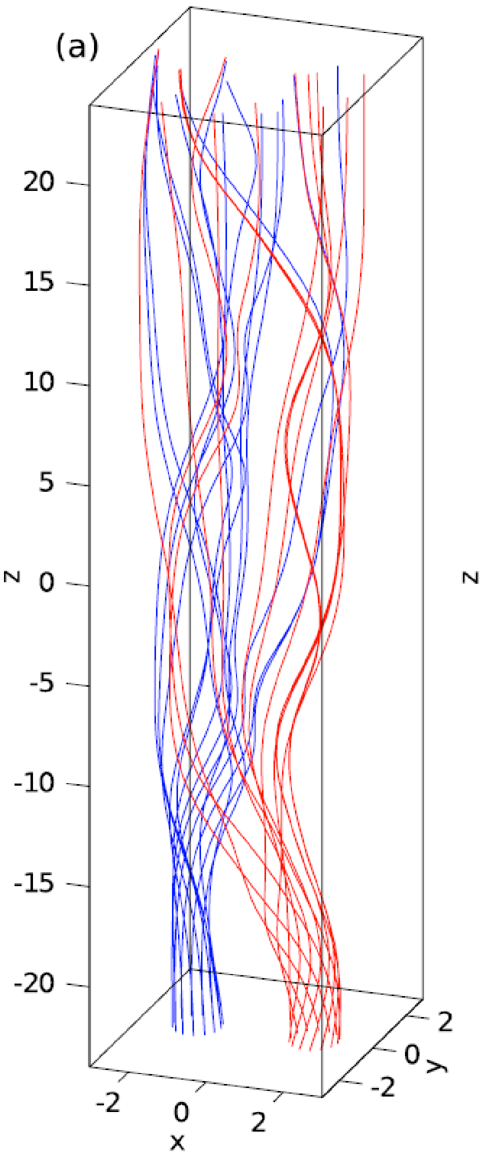
Magnetic Braid Configurations

AAA (trefoil knot)

AABB (Borromean rings)



Fixed Point Index

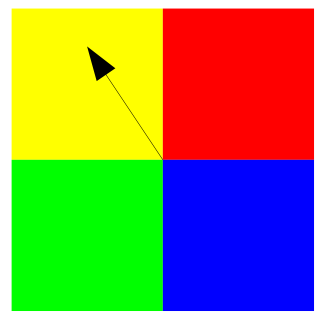


Yeates et al. 2011a

mapping: $(x, y) \rightarrow \mathbf{F}_z(x, y)$

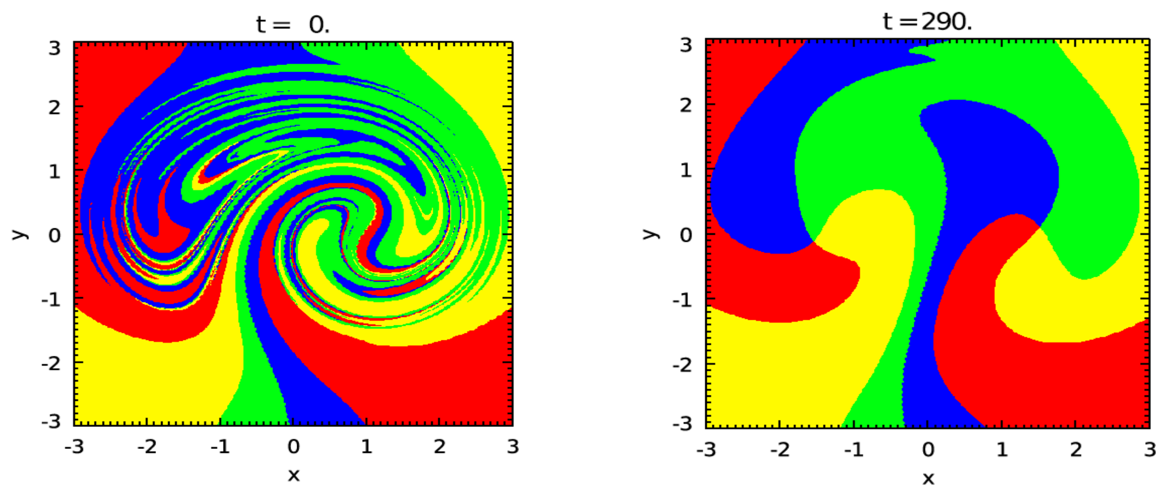
Fixed points: $\mathbf{F}_1(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$

Color coding:



Fixed point index:

$$T = \sum_i t_i \qquad t_i = \pm 1$$



Taylor state is not reached
→ additional constraint

Magnetic Reconnection Rate

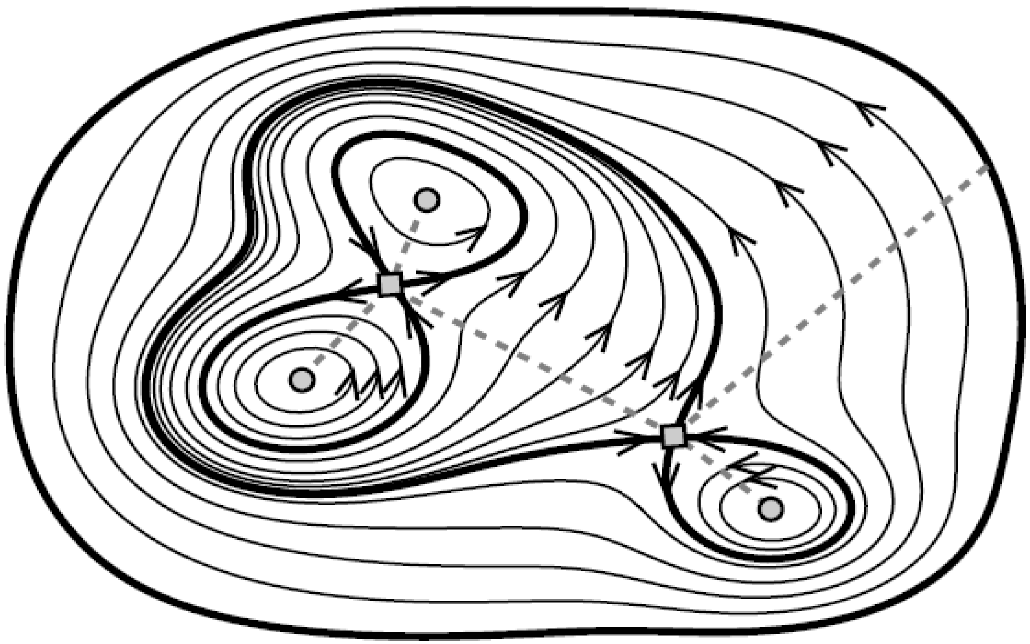
Classic: look for local maxima of $\int \boldsymbol{E} \cdot \boldsymbol{B}$

Partition fluxes 2D:
(Yeates, Hornig 2011b)

$$\boldsymbol{B} = \boldsymbol{\nabla} \times (A\boldsymbol{e}_z)$$

Reconnection rate = magnetic flux through boundaries (spearatrices):

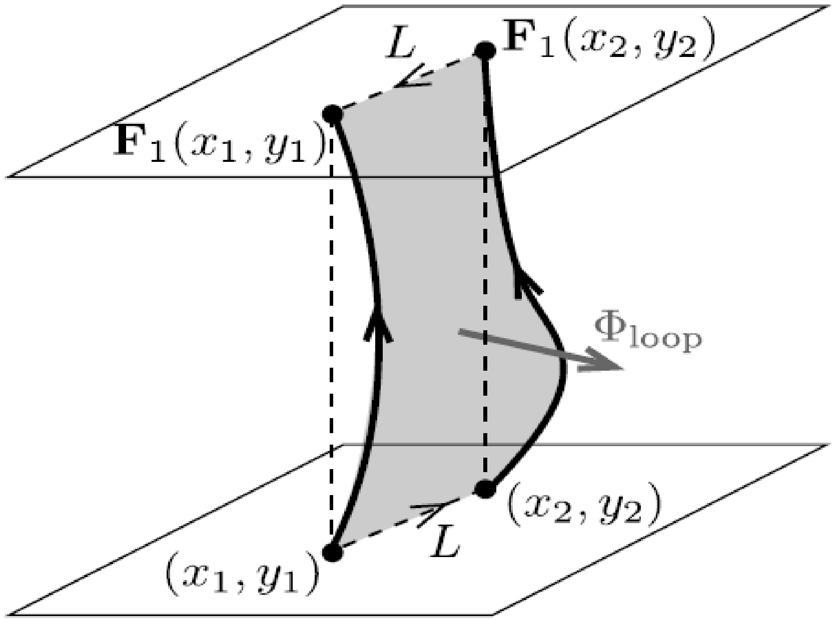
$$\Delta\phi = \sum_i \left| \frac{\mathrm{d}A(\boldsymbol{h}_i)}{\mathrm{d}t} \right|$$



2D Magnetic field.
Thick lines: separatrices.
(Yeates, Hornig 2011b)

Magnetic Reconnection Rate

Partition reconnection rate 3D:
Yeates, Hornig 2011b



Generalized flux function (curly A):

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \boldsymbol{A} \cdot \boldsymbol{B}/B_z \, \mathrm{d}z$$

$$\phi = \int_S \boldsymbol{\nabla} \times \boldsymbol{A} \cdot \, \mathrm{d}\boldsymbol{s} = \int_C \boldsymbol{A} \cdot \, \mathrm{d}\boldsymbol{l}$$

$$\frac{\partial \mathcal{A}}{\partial t} + \boldsymbol{U} \cdot \boldsymbol{\nabla} \mathcal{A} = 0$$

invariant in ideal MHD

Fixed points: $\mathbf{F}_1(x_i, y_i) = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$

Reconnection rate:

$$\Delta\phi = \sum_i \left| \frac{\mathrm{d}\mathcal{A}(\boldsymbol{h}_i)}{\mathrm{d}t} \right|$$

Summary

- Braided magnetic fields are observed in the universe.
- Braiding increases stability through the *realizability condition*.
- Turbulent magnetic field decay is restricted by magnetic helicity.
- Knots and links can be represented as braids.
- Fixed point index as additional constraint in relaxation.
- 'Curly A' as measure for the reconnection rate.

References

Canfield et al. 1999

Canfield, R. C., Hudson, H. S., and McKenzie, D. E.
Sigmoidal morphology and eruptive solar activity.
Geophys. Res. Lett., 26:627, 1999

Leka et al., 1996

Leka, K. D., Caneld, R. C., McClymont, A. N., and van Driel-Gesztelyi, L.,
Evidence for Current-carrying Emerging Flux.
Astrophysical Journal, 462:547.

Gibson et al., 2002

Gibson, S. E., Fletcher, L., Zanna, G. D., et al.,
The structure and evolution of a sigmoidal active region.
The Astrophysical Journal, 574:1021

Woltjer 1958

Woltjer, L.
A Theorem on Force-Free Magnetic Fields.
Proceedings of the National Academy of Sciences of the United States of America, 44:489, 1958

References

Taylor 1974

Taylor, J. B.
Relaxation of Toroidal Plasma and Generation of Reverse Magnetic Fields.
Physical Review Letters, 33:1139, 1974

Pouquet et al., 1976

Pouquet, A., Frisch, U., and Leorat, J.,
Strong MHD helical turbulence and the nonlinear dynamo effect.
Journal of Fluid Mechanics, 77:321, 1976.

Leorat et al., 1975

Leorat, J., Frisch, U., and Pouquet, A.
Helical magnetohydrodynamic turbulence and the nonlinear dynamo problem.
In V. Canuto, editor, Role of Magnetic Fields in Physics and Astrophysics, volume 257 of New York Academy Sciences Annals, pages 173-176, 1975

Ruzmaikin and Akhmetiev 1994

A. Ruzmaikin and P. Akhmetiev.
Topological invariants of magnetic fields, and the effect of reconnections.
Phys. Plasmas, vol. 1, pp. 331–336, 1994.

References

Del Sordo et al. 2010

Fabio Del Sordo, Simon Candelaresi, and Axel Brandenburg.
Magnetic-field decay of three interlocked flux rings with zero linking number.
Phys. Rev. E, 81:036401, Mar 2010.

Candelaresi and Brandenburg 2011

Simon Candelaresi, and Axel Brandenburg.
Decay of helical and non-helical magnetic knots.
Phys. Rev. E, 84:016406, 2011

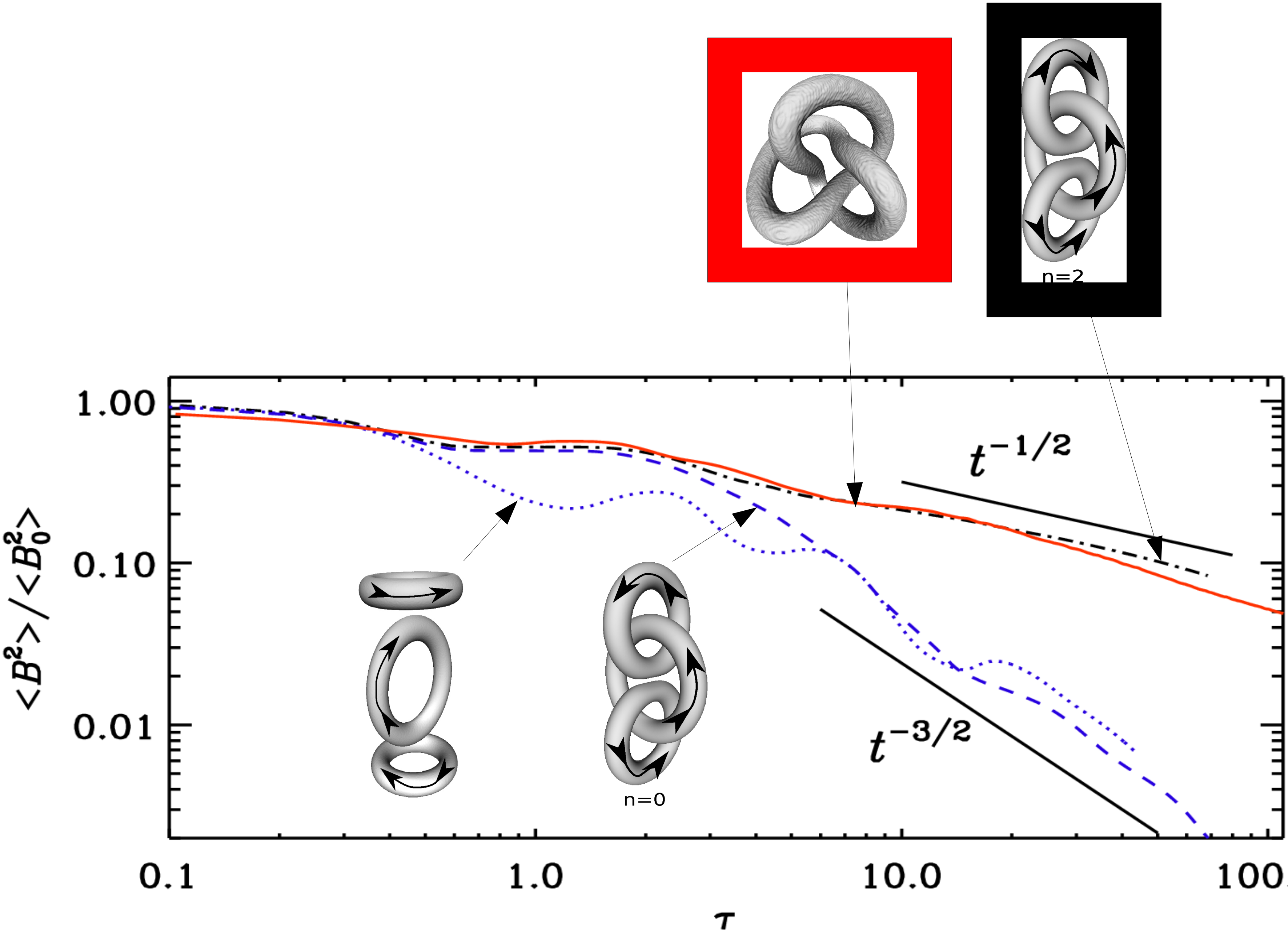
Yeates et al. 2011a

Yeates, A. R., Hornig, G. and Wilmot-Smith, A. L.
Topological Constraints on Magnetic Relaxation.
Phys. Rev. Lett. 105, 085002, 2010

Yeates, Hornig 2011b

Yeates, A. R., and Hornig, G.,
A generalized flux function for three-dimensional magnetic reconnection.
Physics of Plasmas, 18:102118, 2011

Magnetic energy decay



Simulations

- 256^3 mesh point
- Isothermal compressible gas
- Viscous medium
- Periodic boundaries

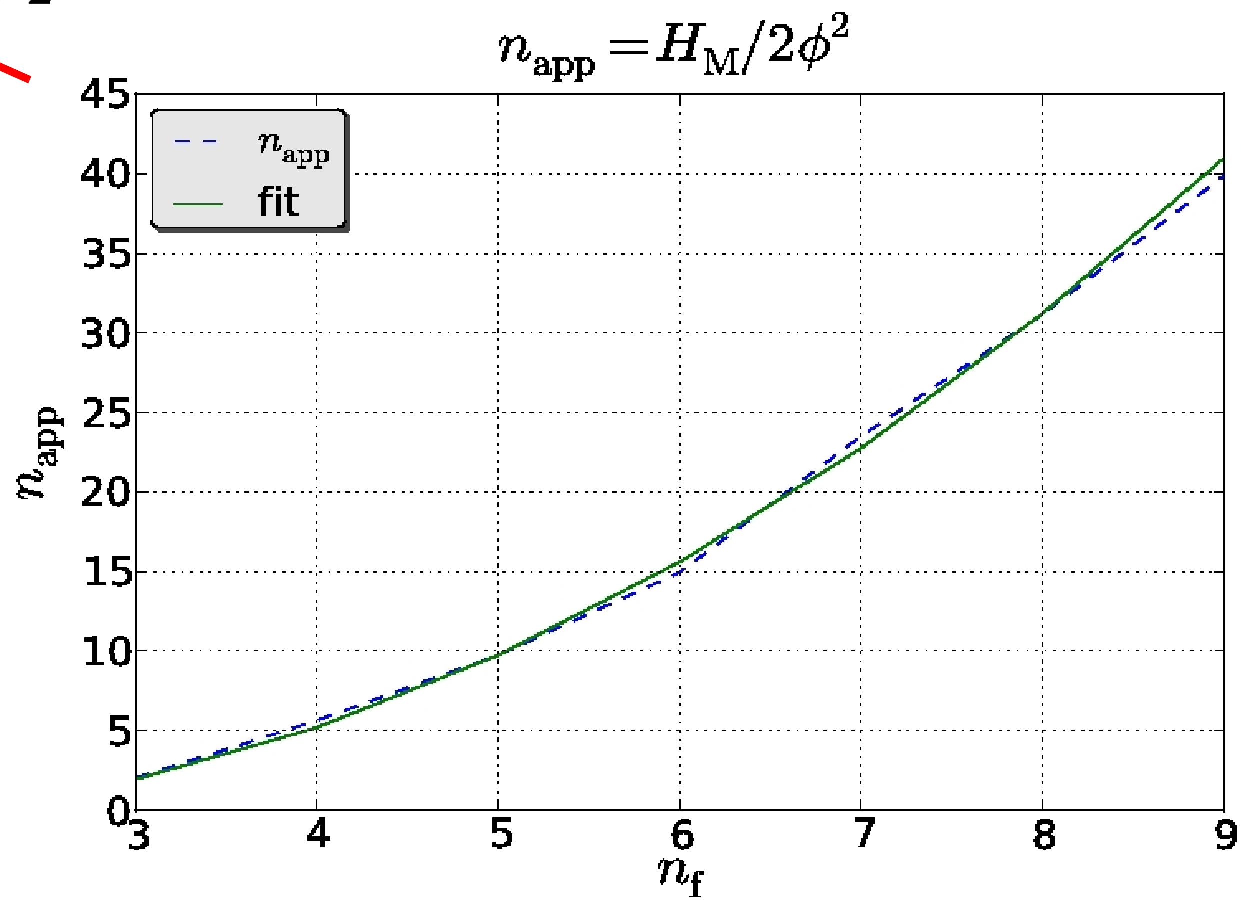
$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \eta \nabla^2 \mathbf{A}$$

$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -c_\mathrm{S}^2 \boldsymbol{\nabla} \ln \rho + \mathbf{J} \times \mathbf{B}/\rho + \mathbf{F}_\mathrm{visc}$$

$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{U}$$

N-foil Knots

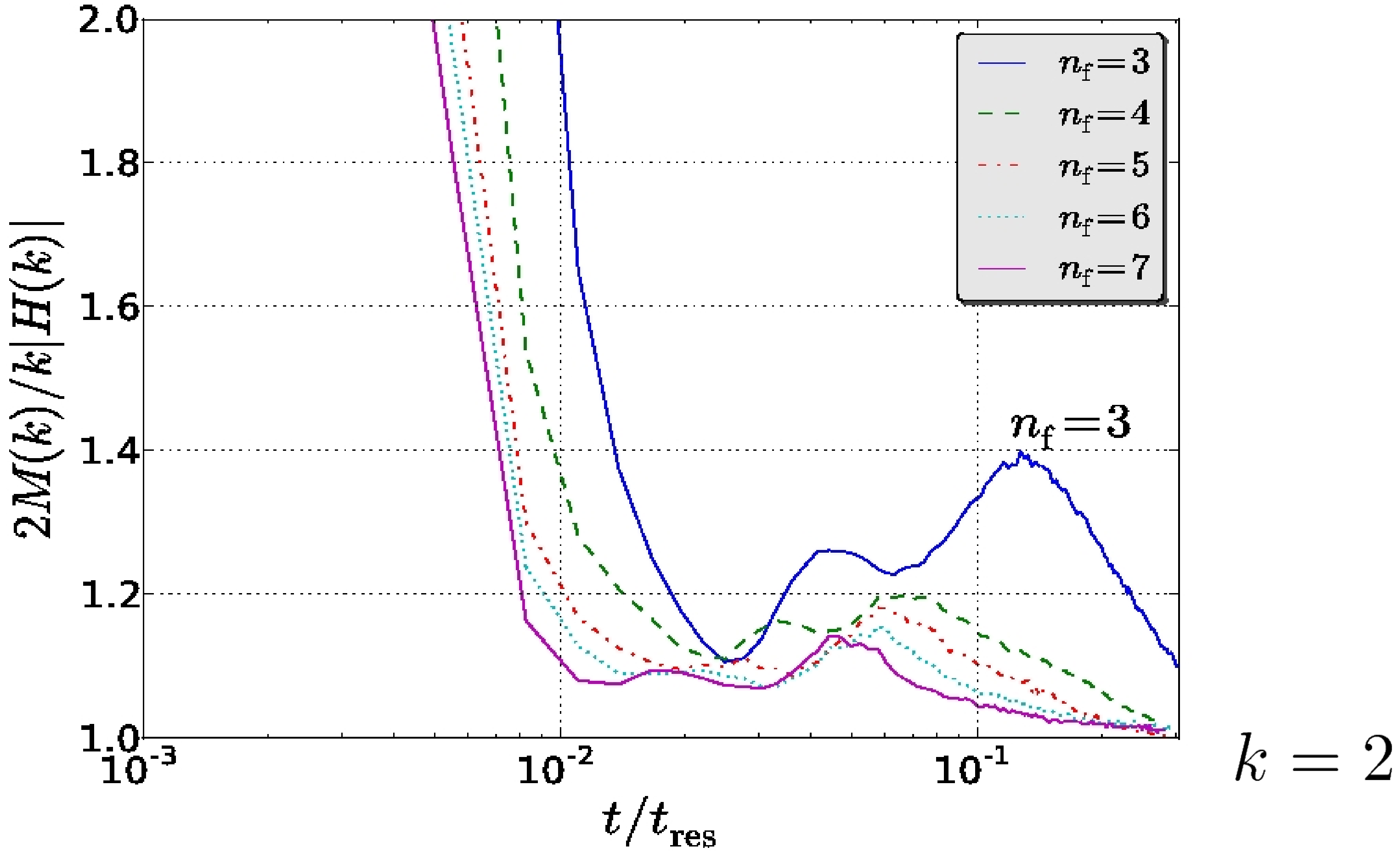
~~$H_\mathrm{M} = 2n\phi_1\phi_2$~~



$$H_\mathrm{M} = (n_\mathrm{f} - 2)n_\mathrm{f}\phi^2/2$$

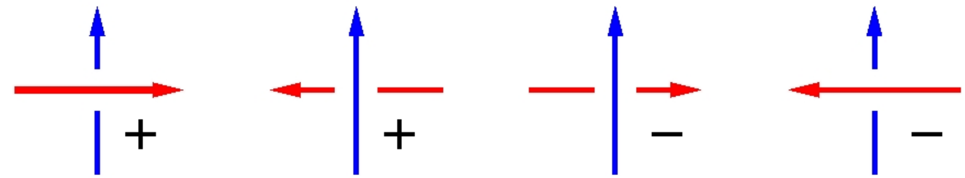
N-foil Knots

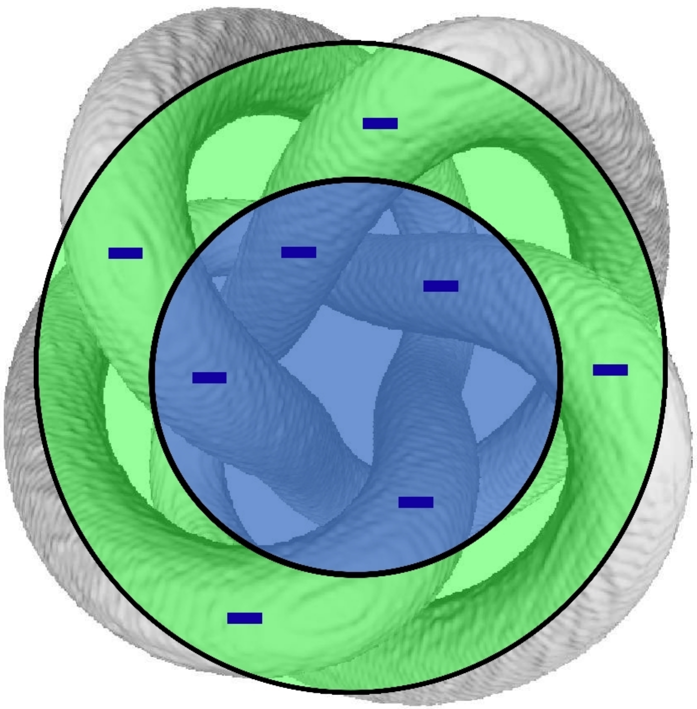
$$2M(k)/(|H(k)|k)$$



Realizability condition more important for high $n_{\rm f}$.

Linking Number





$$n_{\text{linking}} = (n_+ - n_-)/2$$

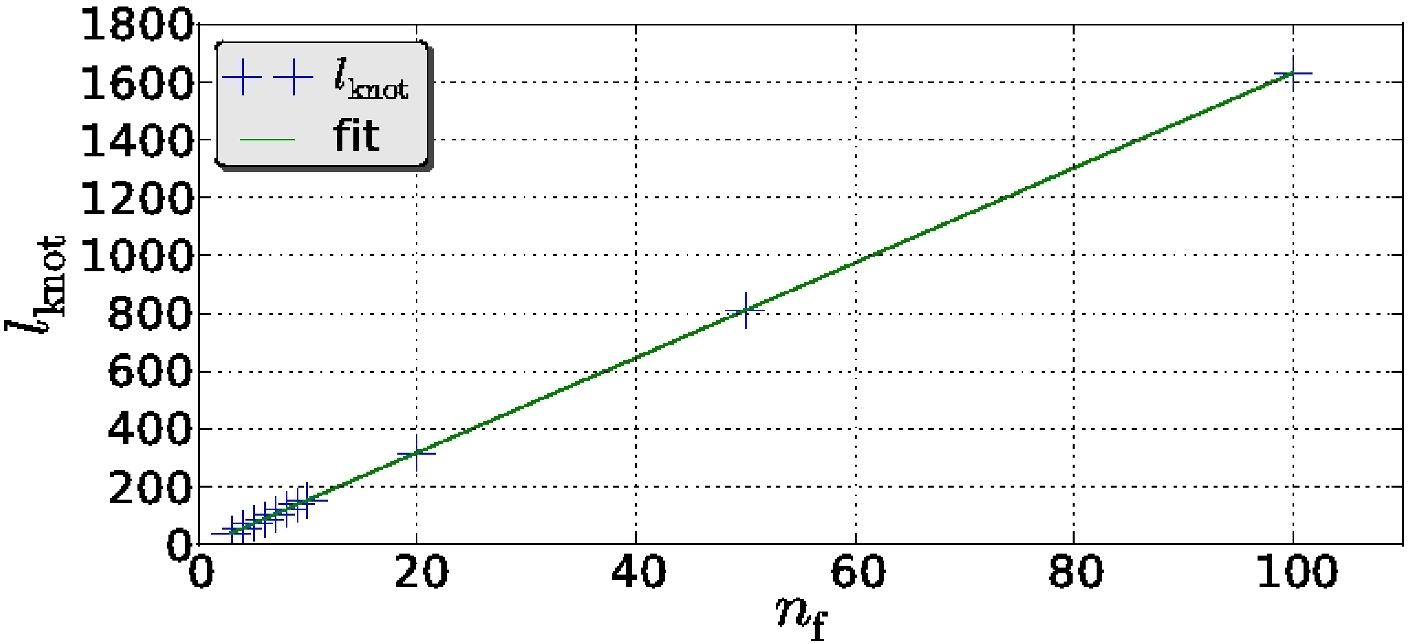
Number of crossings increases like n_{f}^2

$$H_{\text{M}} \propto n_{\text{linking}}$$

$$H_{\text{M}} \propto n_{\text{f}}^2$$

Sign of the crossings for the 4-foil knot

Helicity vs. Energy



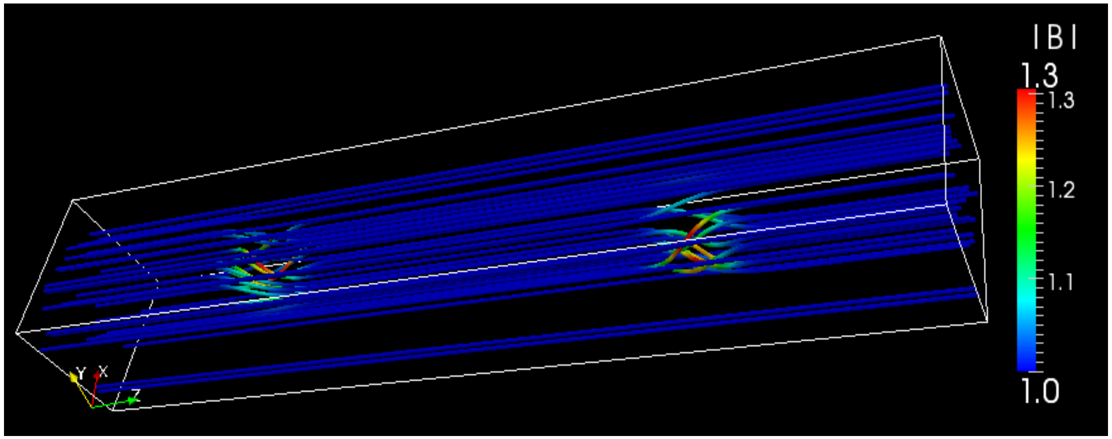
$$E_{\rm M} \propto l_{\rm knot} \propto n_{\rm f}$$

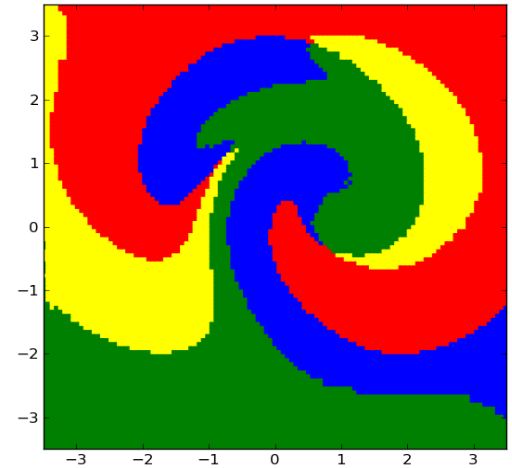
$$H_{\rm M} \propto n_{\rm f}^2$$

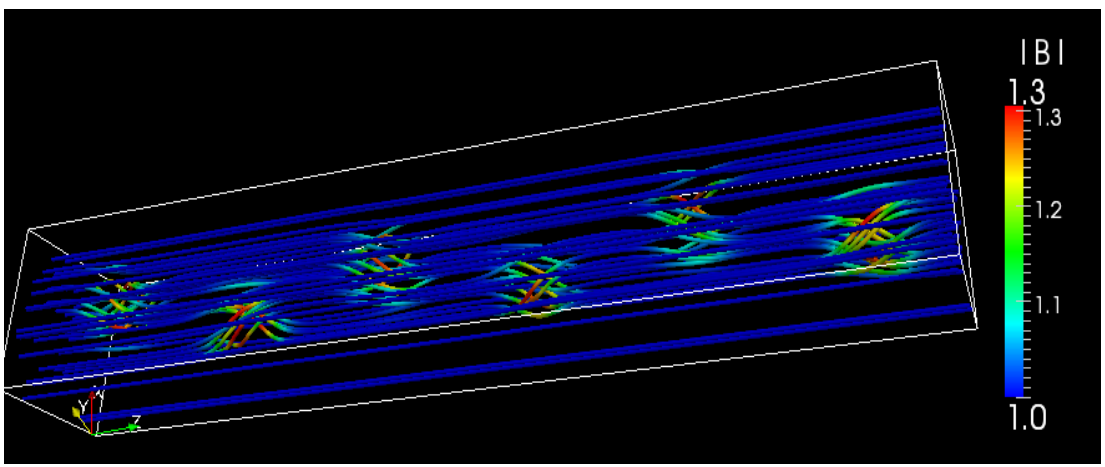
Knot is more strongly packed with increasing .

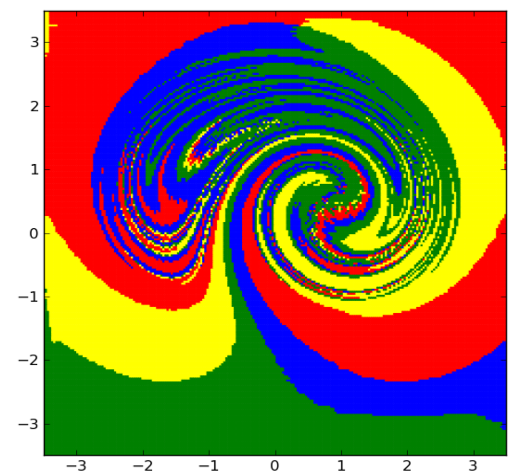
Magnetic energy is closer to its lower limit for high .

Field Line Tracing









Generalized flux function:

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

Reconnection rate:

$$\sum_i \frac{\mathrm{d}\mathcal{A}(\mathbf{x}_i)}{\mathrm{d}t}$$