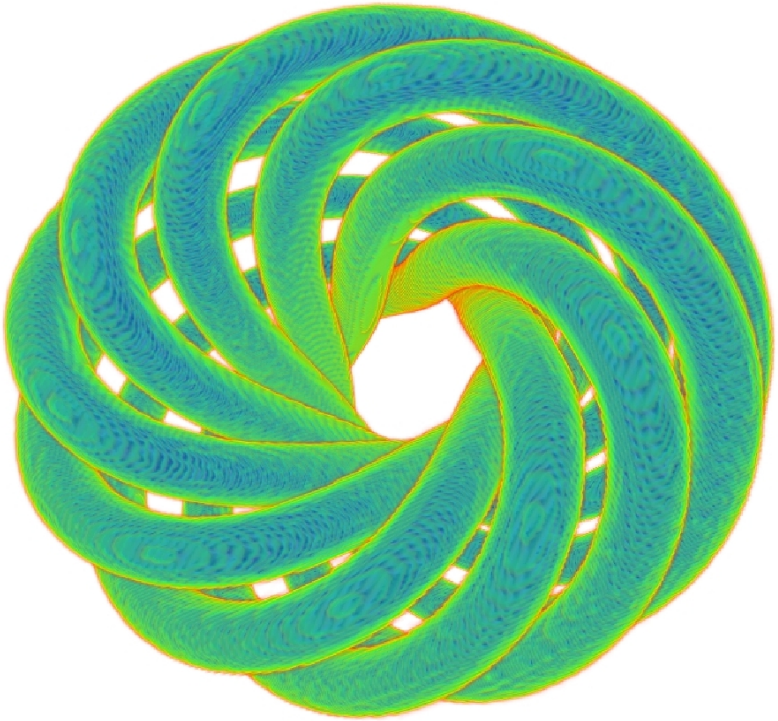
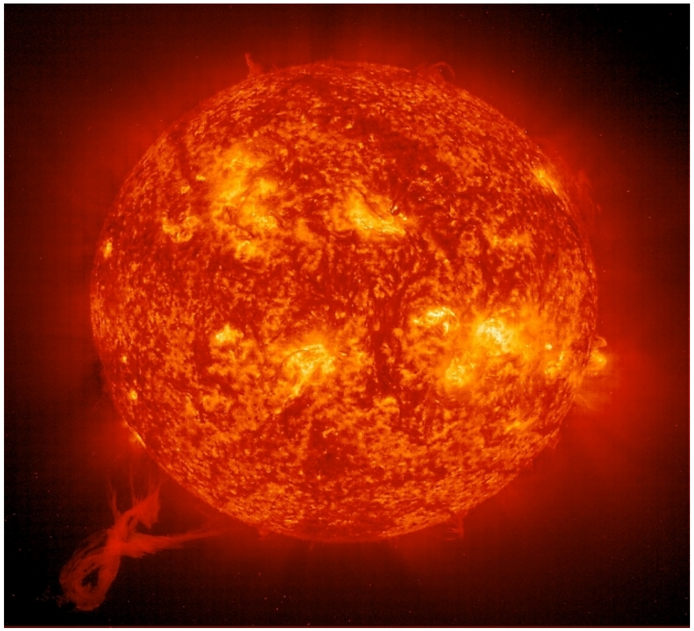
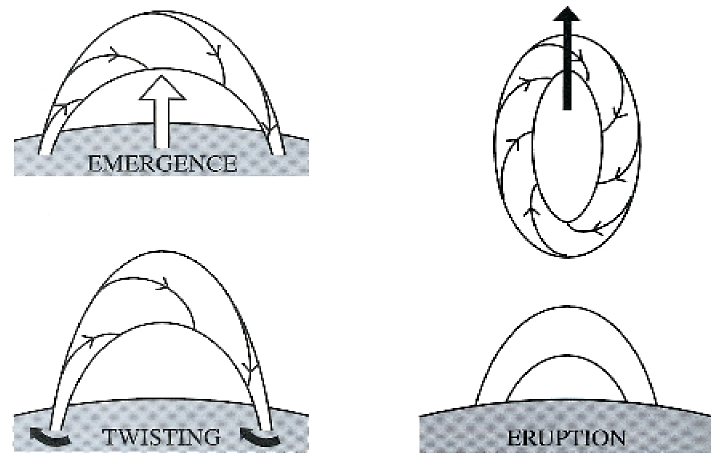
Topological constraints in magnetic field relaxation

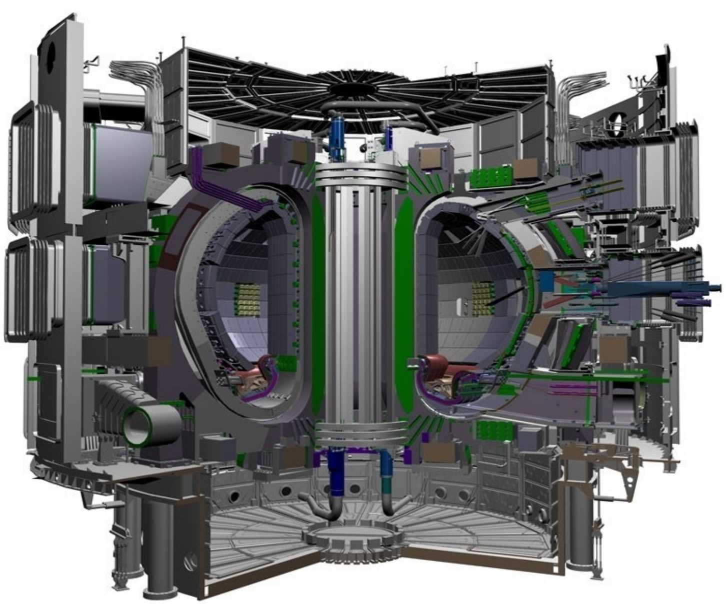
Simon Candelaresi



Twisted Magnetic Fields







- Twisted fields are more likely to erupt *(Canfield et al. 1999)*.
- Twist increases the stability of magnetic fields in tokamaks.

Twisted Field in the Sun

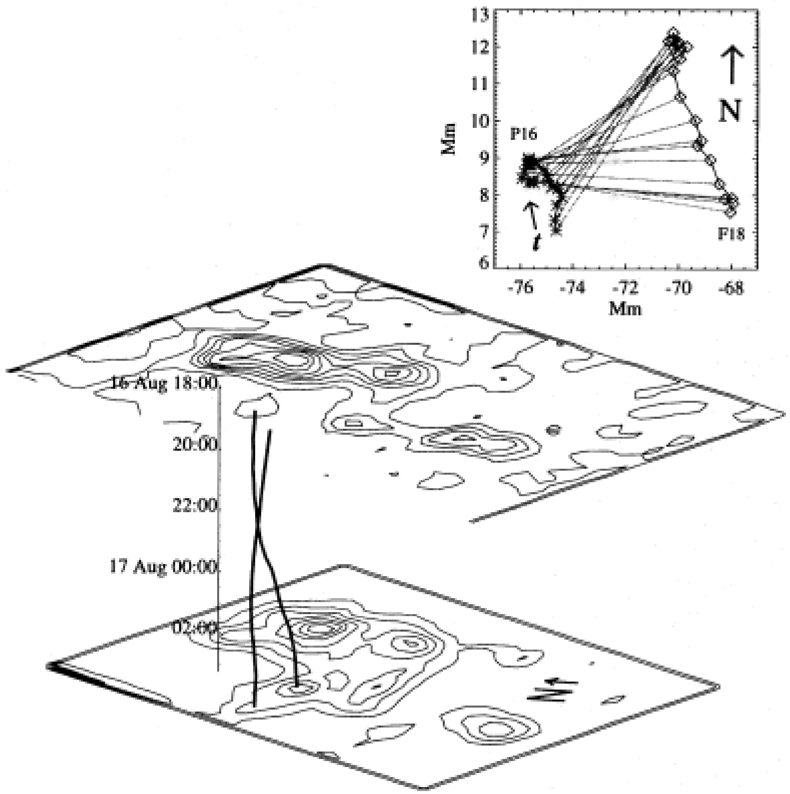
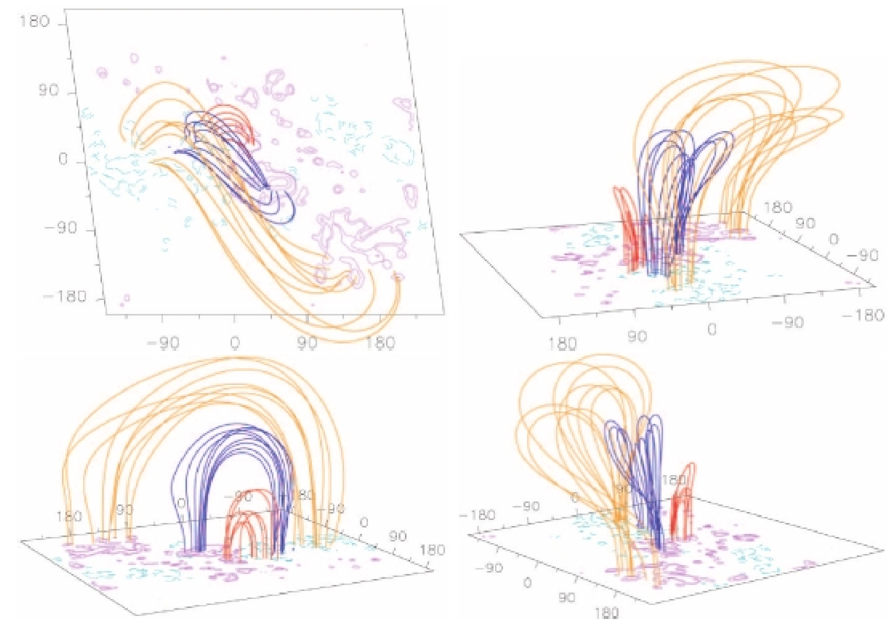


FIG. 3b

Magnetic bipoles' movement on the Sun's surface.
(Leka et al. 1996)



Force-free extrapolation of the photospheric magnetic field from 1999, August 21.
(Gibson et al. 2002)

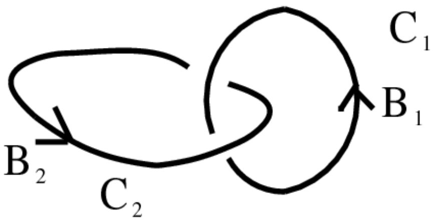
Force free condition:

$$\nabla \times B = \alpha B$$
$$J \times B = 0$$

Magnetic Helicity

$$H_{\mathrm{M}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 2n\phi_1\phi_2$$

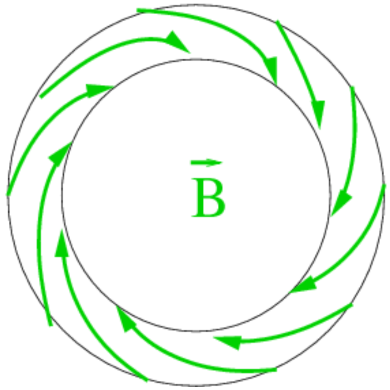
$$\phi_i = \int_{S_i} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$



Realizability condition:

$$E_{\mathrm{m}}(k) \geq k|H(k)|/2\mu_0$$

Magnetic energy is bound from below by magnetic helicity.



twisted field

magnetic helicity conservation

$$\mathrm{Re}_{\mathrm{M}} \to \infty$$

$$\frac{\mathrm{d}H_{\mathrm{M}}}{\mathrm{d}t} = 0$$

trefoil knot

Stability Criteria

Ideal MHD: $\eta = 0$

$\Downarrow$

Induction equation: $\dfrac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B})$

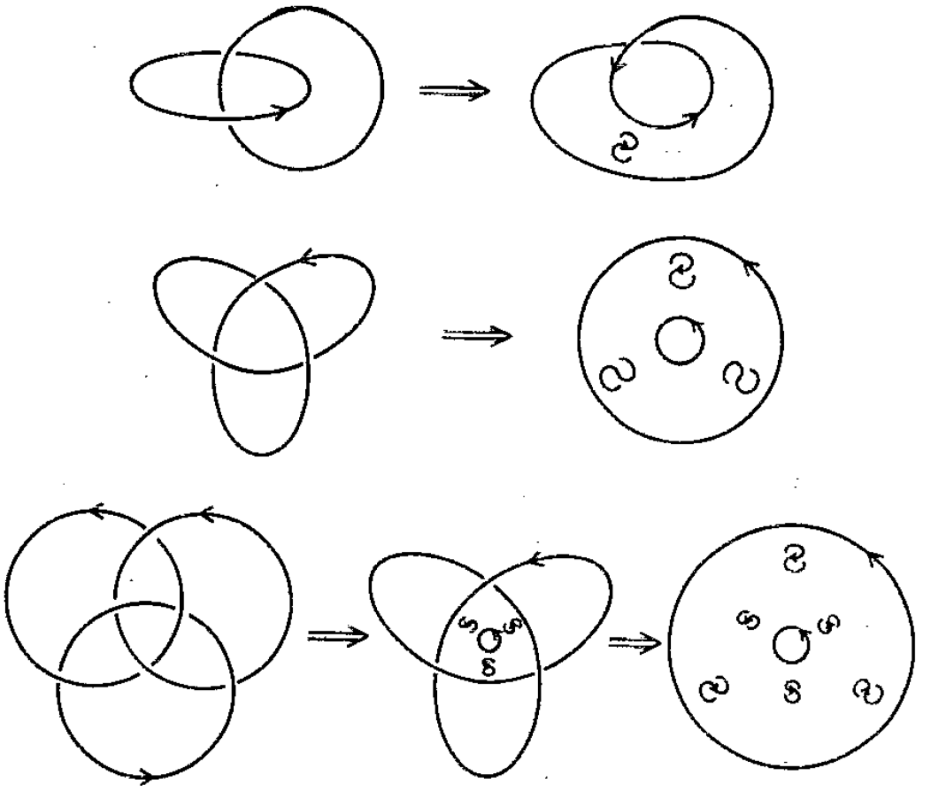
	constraint	equilibrium
Woltjer (1958):	$\dfrac{\partial}{\partial t} \displaystyle\int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha \boldsymbol{B}$
Taylor (1974):	$\dfrac{\partial}{\partial t} \displaystyle\int_{\tilde{V}} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha(a, b) \boldsymbol{B}$

$\alpha(a, b)$: constant along field line

V total volume $\quad \tilde{V}$ volume along magnetic field line

Reconnection Characteristics

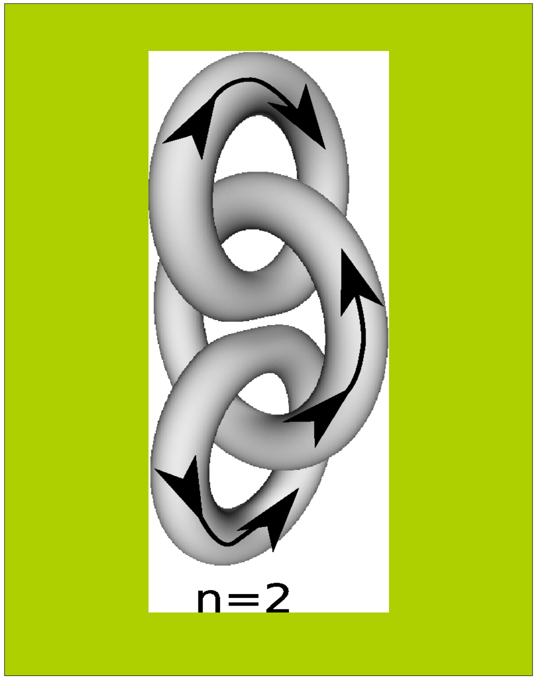
Conversion of linking into twisting:



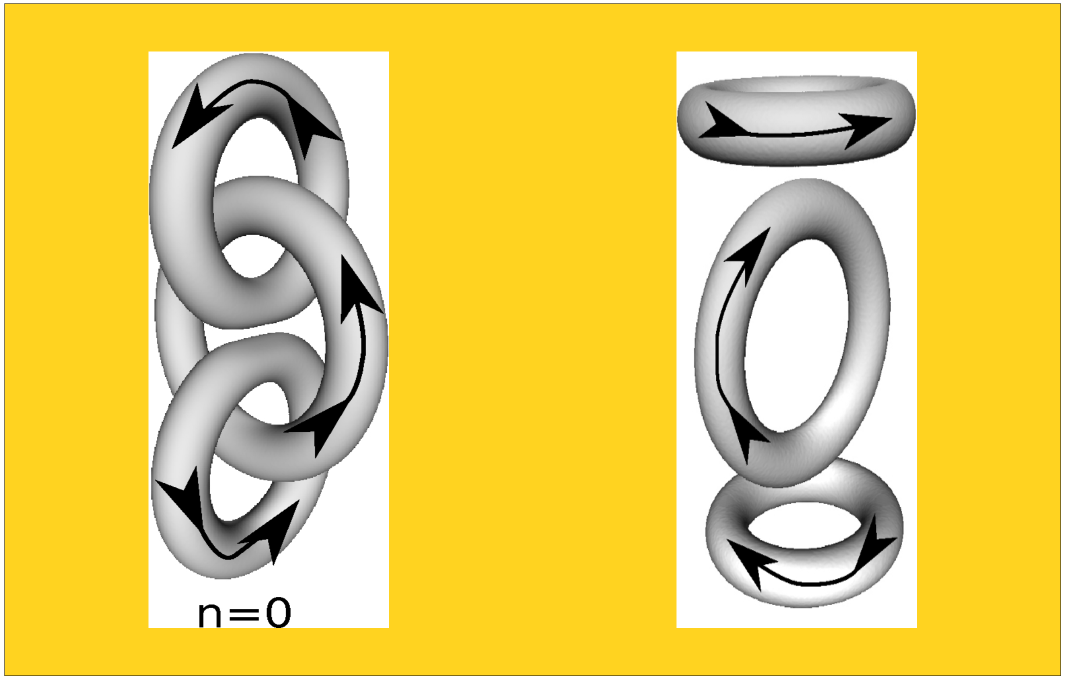
Ruzmaikin and Akhmetiev, 1994

Interlocked Flux Rings

$H_{\mathrm{M}} \neq 0$



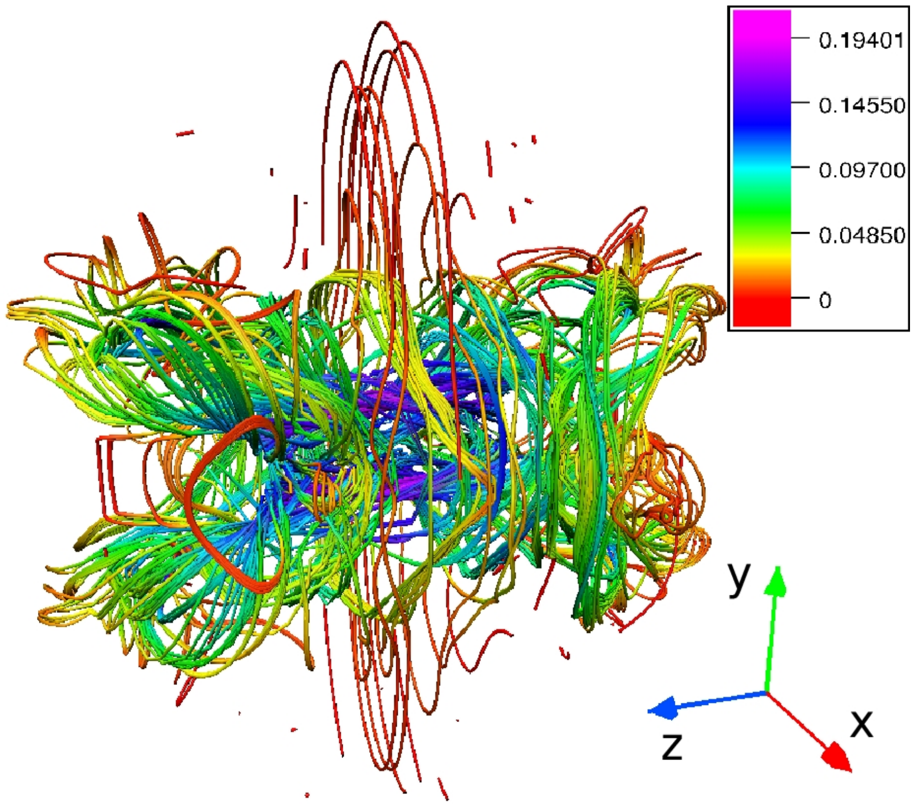
$H_{\mathrm{M}} = 0$



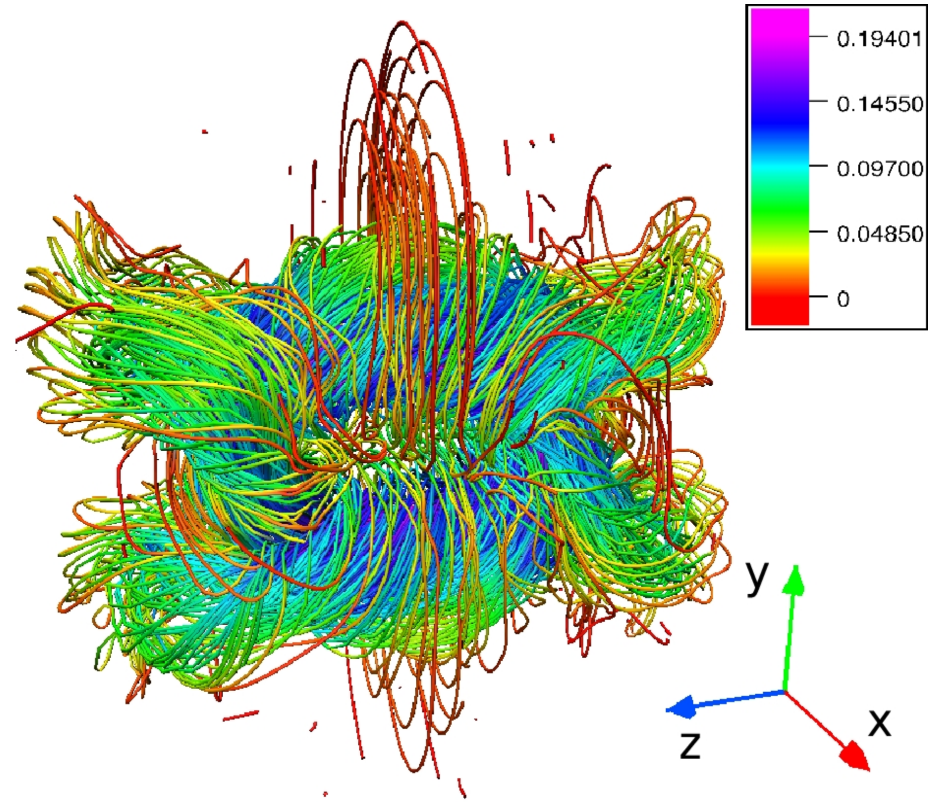
- isothermal compressible gas
- viscous medium
- periodic boundaries

Interlocked Flux Rings

$\tau = 4$

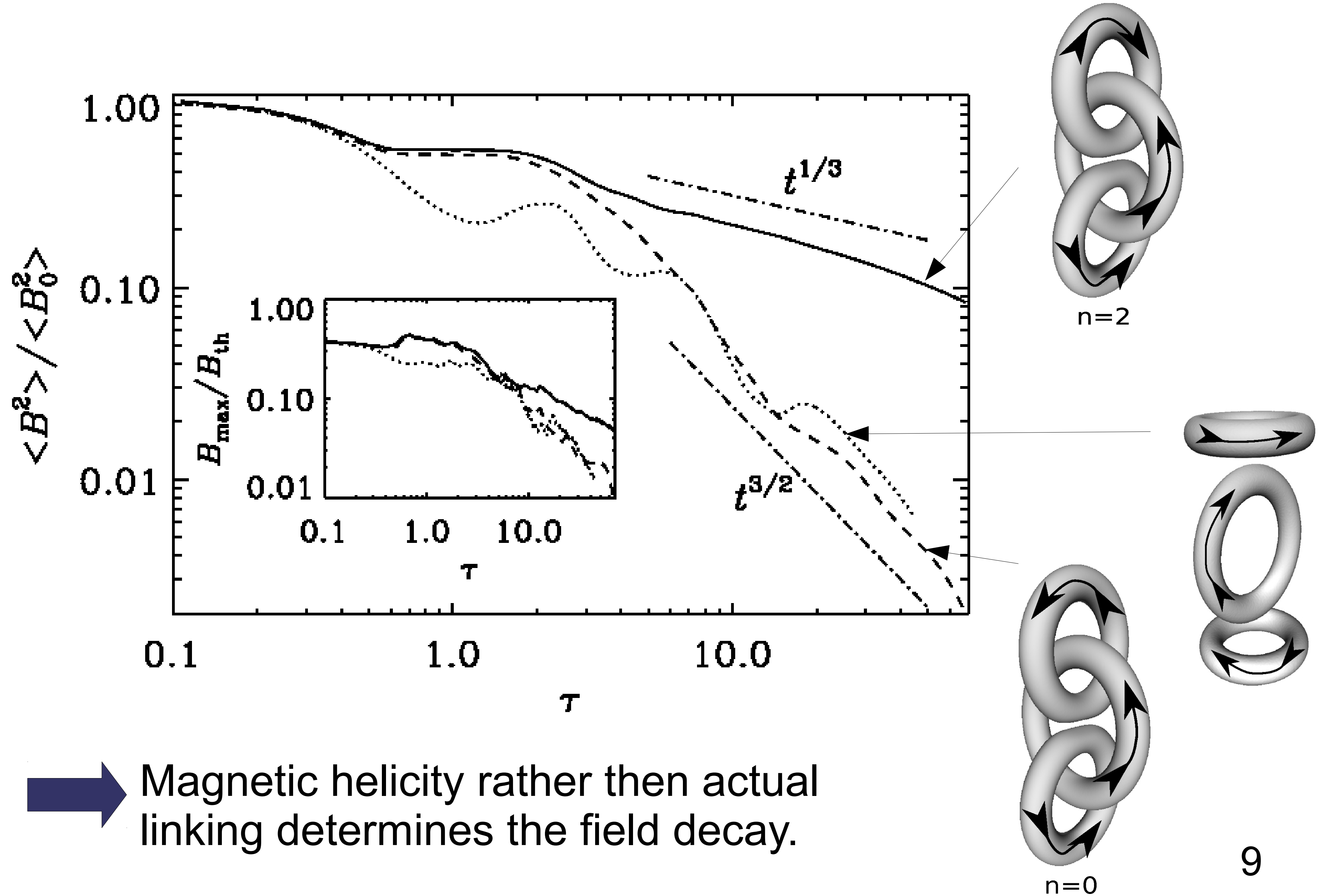


$H_{\mathrm{M}} = 0$



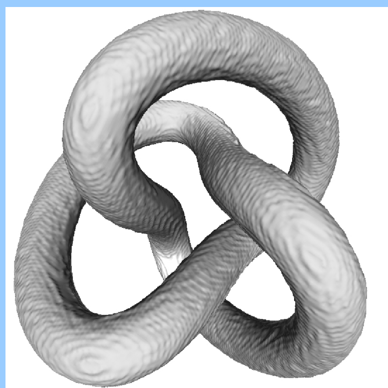
$H_{\mathrm{M}} \neq 0$

Interlocked Flux Rings

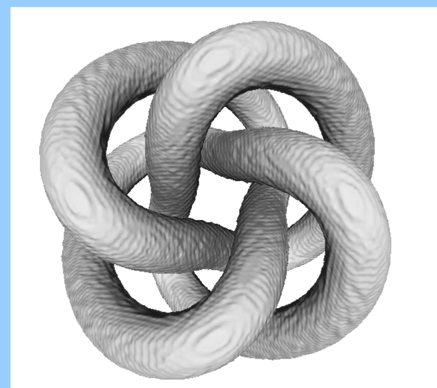


Magnetic helicity rather then actual linking determines the field decay.

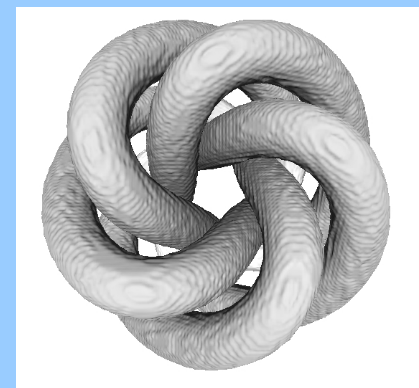
N-foil Knots



3-foil

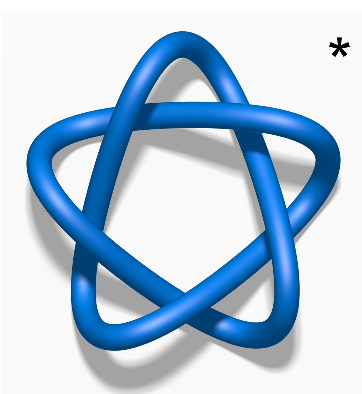


4-foil

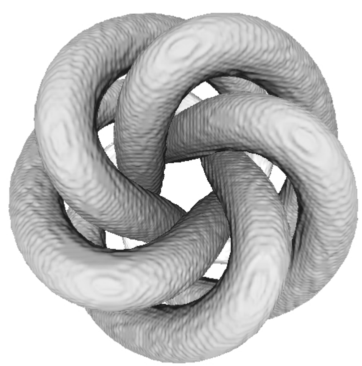


5-foil

6-foil 7-foil



$\neq$

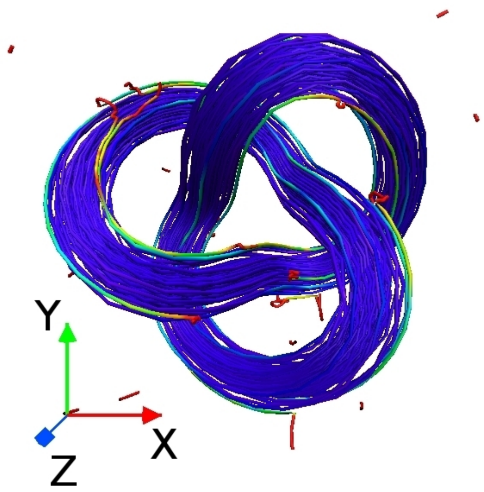


$$x(s) = \begin{pmatrix} (C + \sin s n_\mathrm{f}) \sin[s(n_\mathrm{f} - 1)] \\ (C + \sin s n_\mathrm{f}) \cos[s(n_\mathrm{f} - 1)] \\ D \cos s n_\mathrm{f} \end{pmatrix}$$

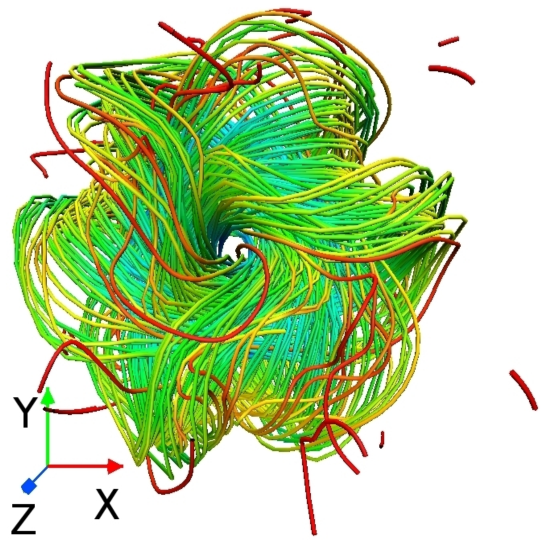
cinquefoil knot

* from Wikipedia, author: Jim.belk

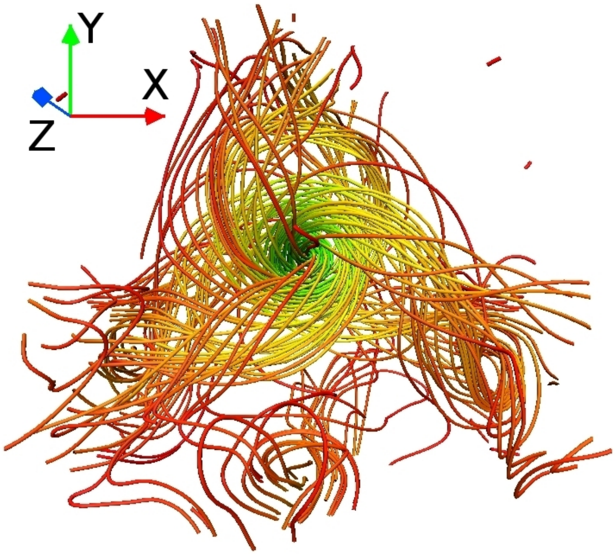
N-foil Knots



$t = 0$



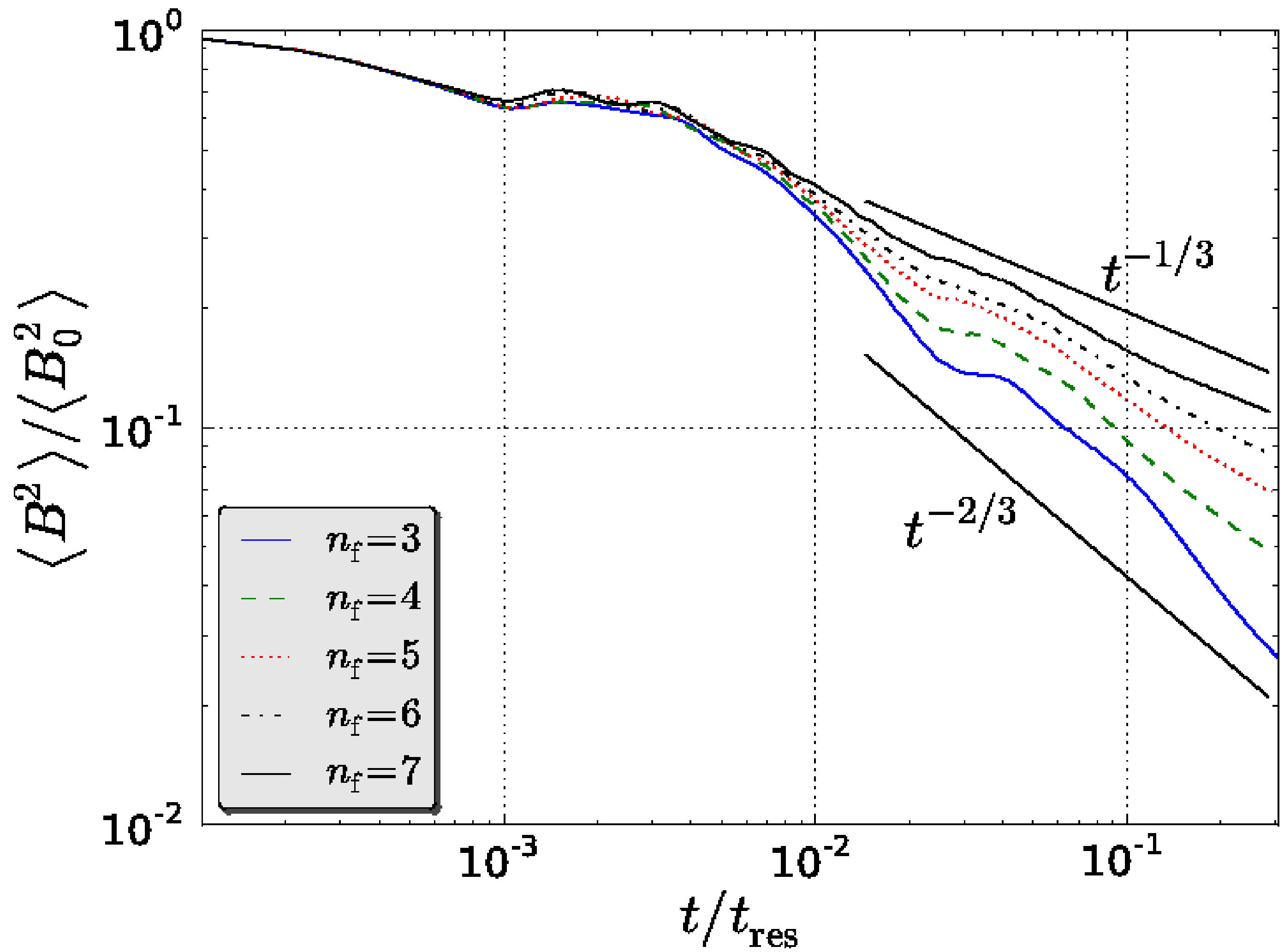
$t = 6$



$t = 39$

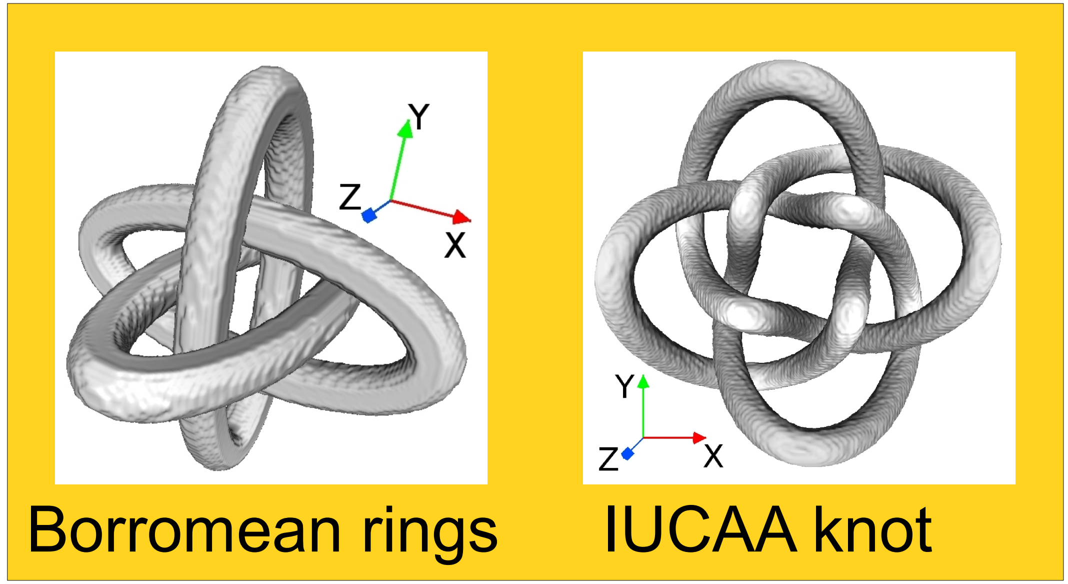
- Magnetic helicity is approximately conserved.
- Self-linking is transformed into twisting after reconnection.

N-foil Knots



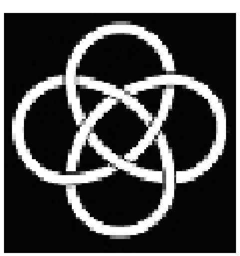
Slower decay for higher $n_{\rm f}$.

IUCAA Knot and Borromean Rings



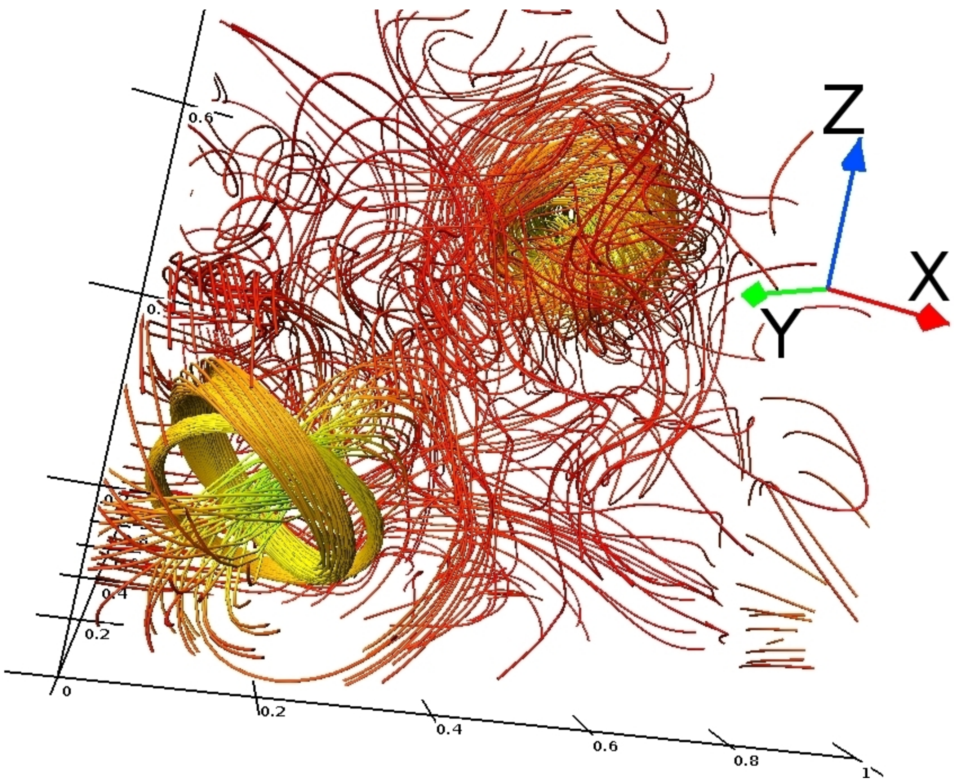
$$H_{\mathrm{M}} = 0$$

- Is magnetic helicity sufficient?
- Higher order invariants?

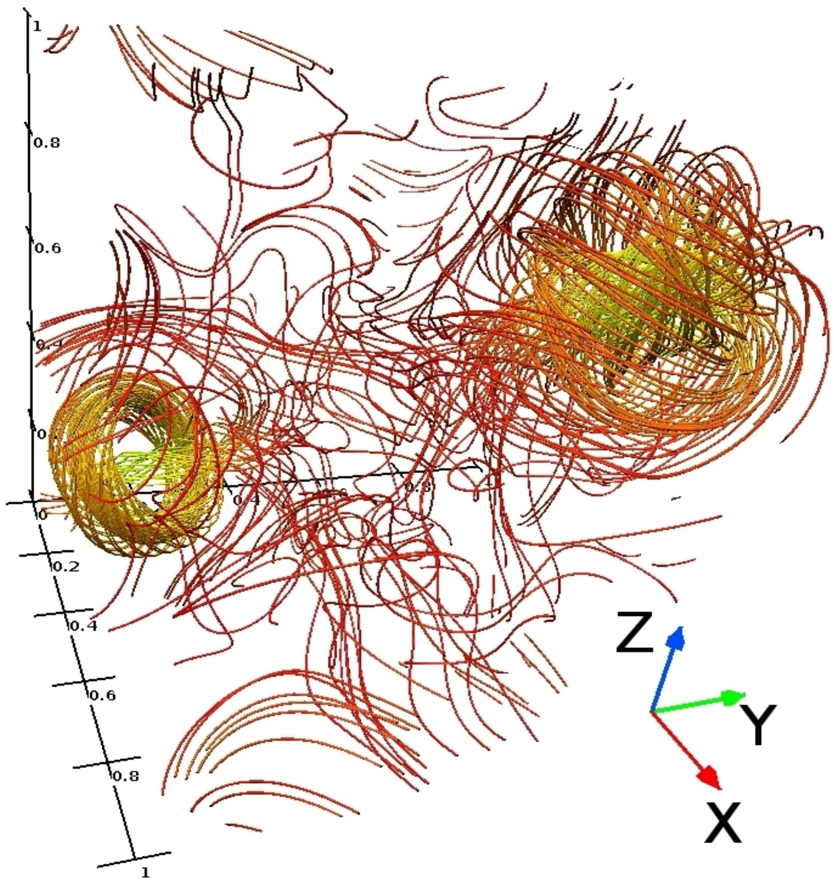


IUCAA = The Inter-University Centre for Astronomy and Astrophysics, Pune, India

Reconnection Characteristics



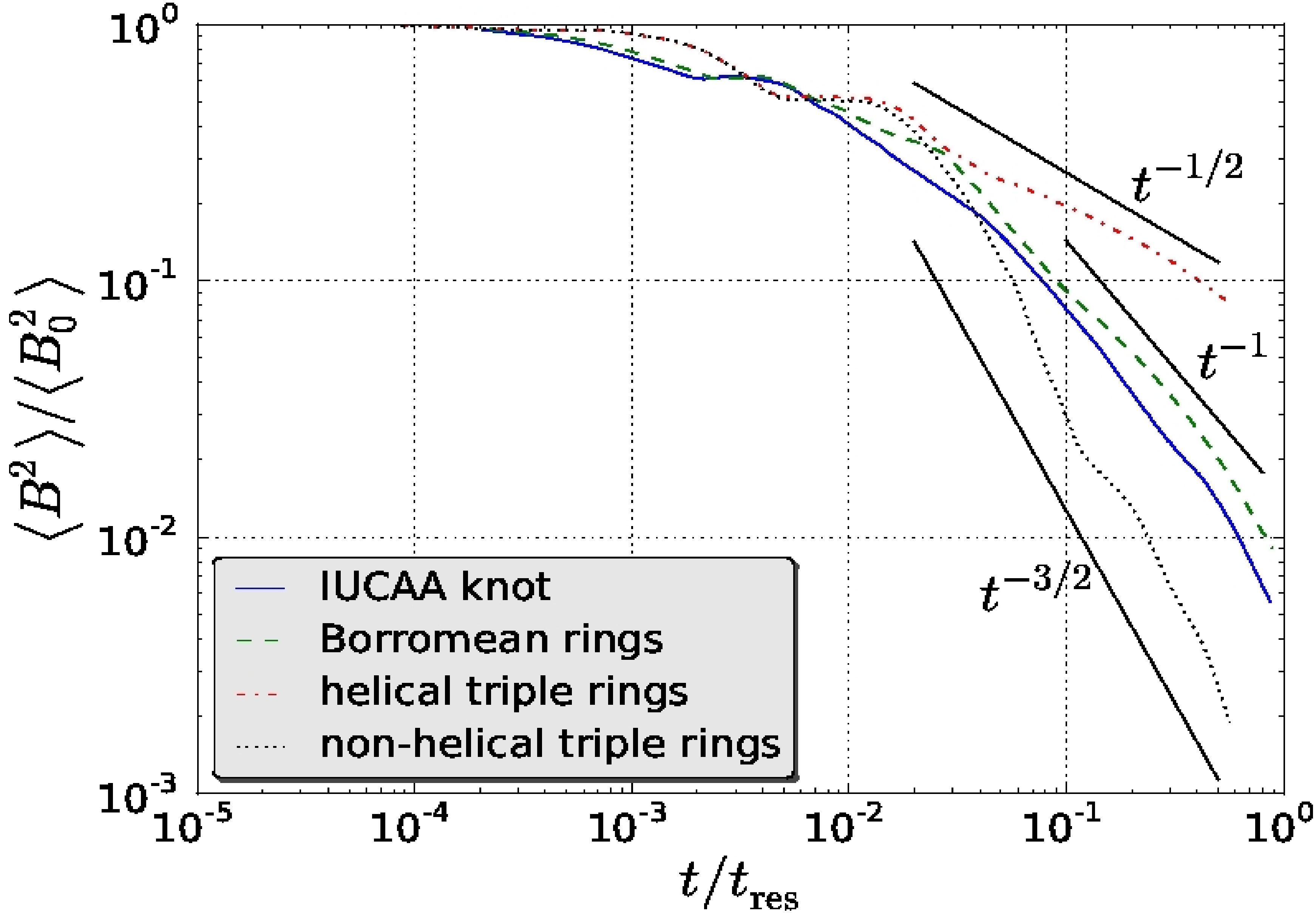
$t = 70$



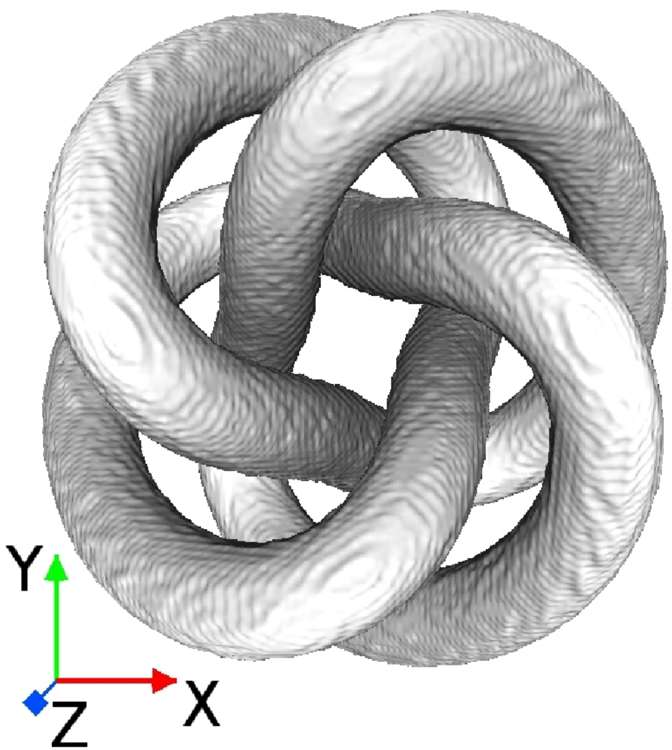
$t = 78$

3 rings → Twisted ring + interlocked rings → 2 twisted rings

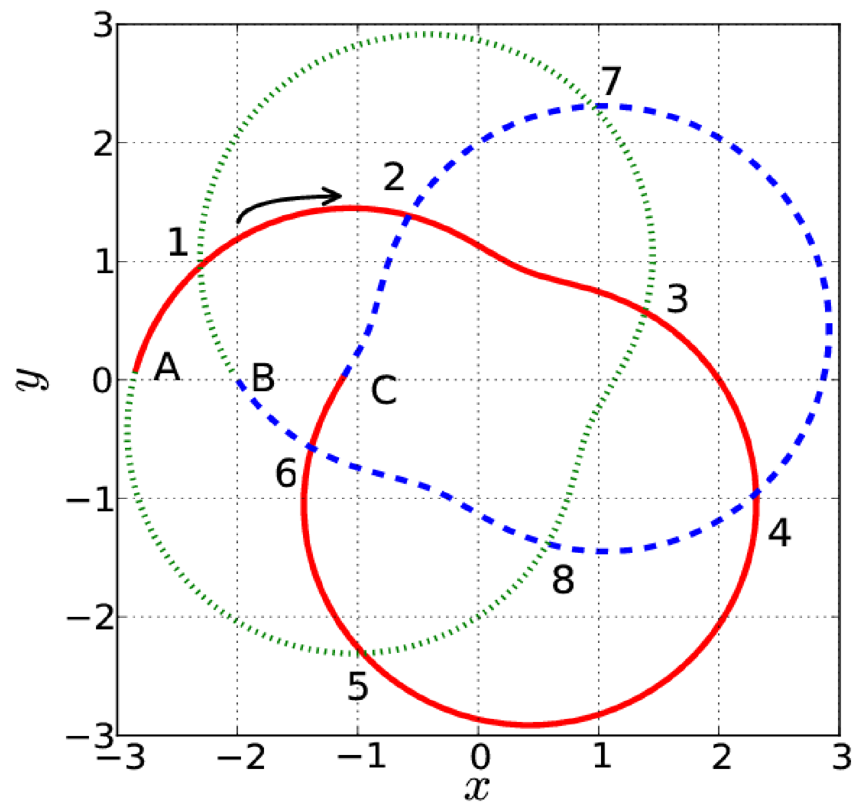
Magnetic Energy Decay

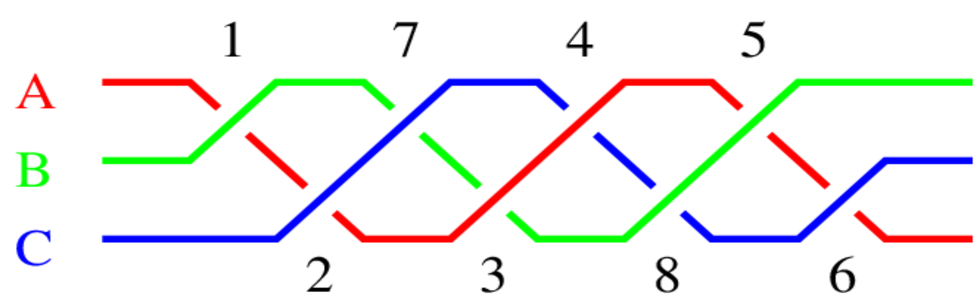


Braid Representation



4-foil knot



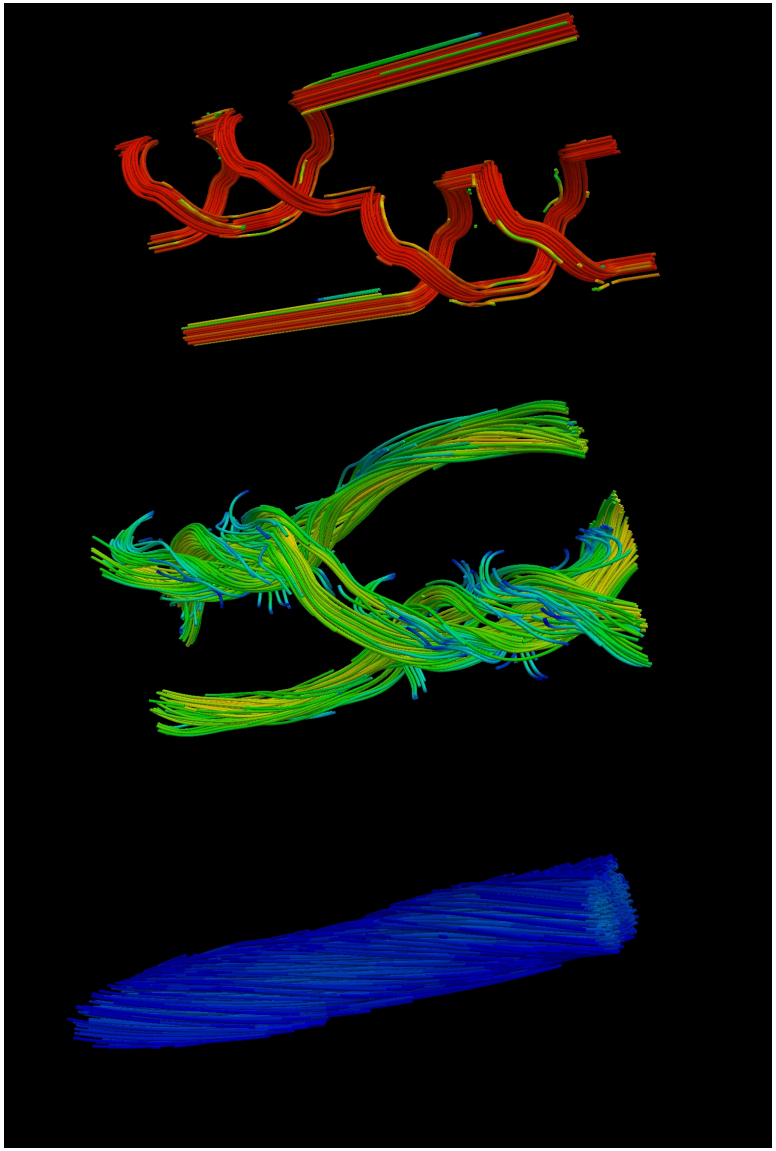


Word: ABABABAB $B_z > 0$

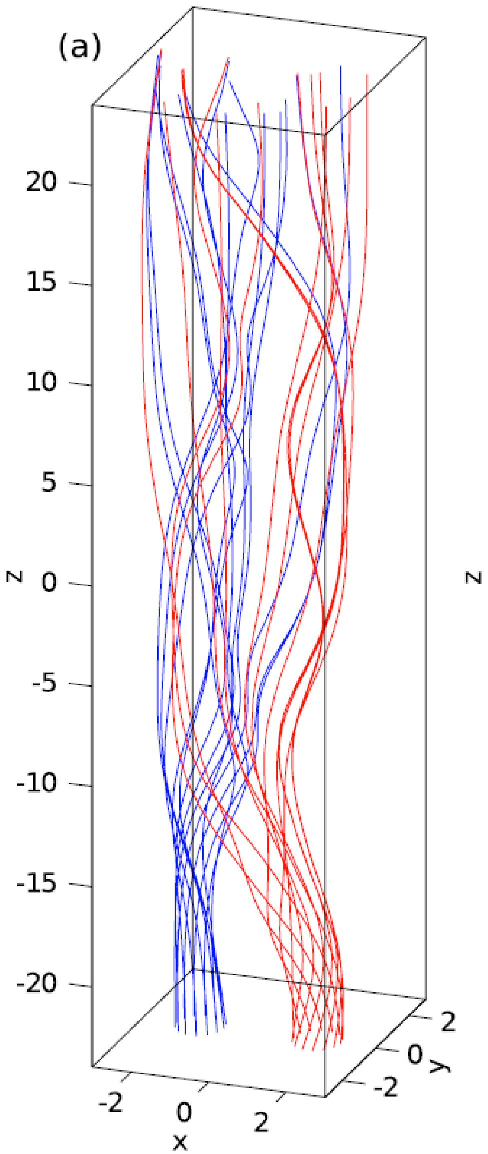
Magnetic Braid Configurations

AAA (trefoil knot)

AABB (Borromean rings)



Fixed Point Index

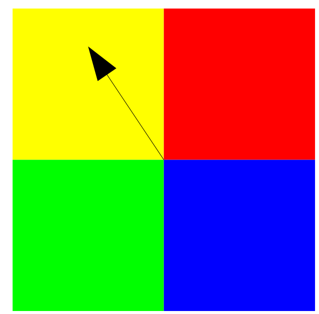


Yeates et al. 2011a

mapping: $(x, y) \rightarrow \mathbf{F}_z(x, y)$

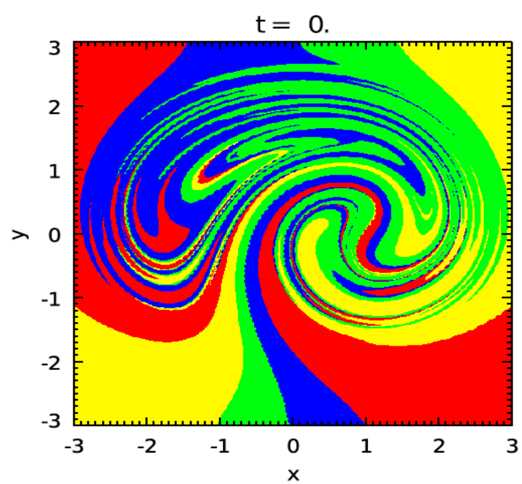
Fixed points: $\mathbf{F}_1(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$

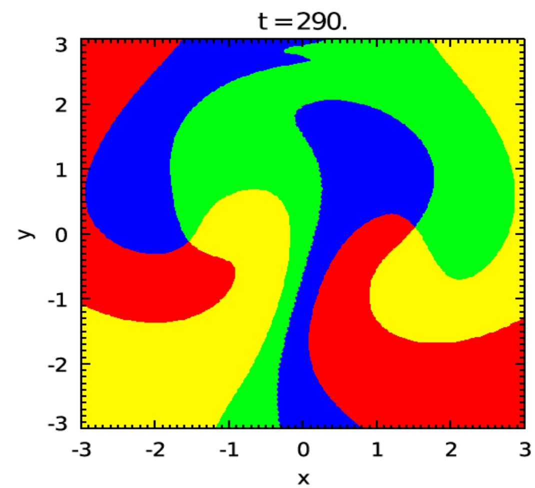
Color coding:



Fixed point index:

$$T = \sum_i t_i \qquad t_i = \pm 1$$





Taylor state is not reached
→ additional constraint

Magnetic Reconnection Rate

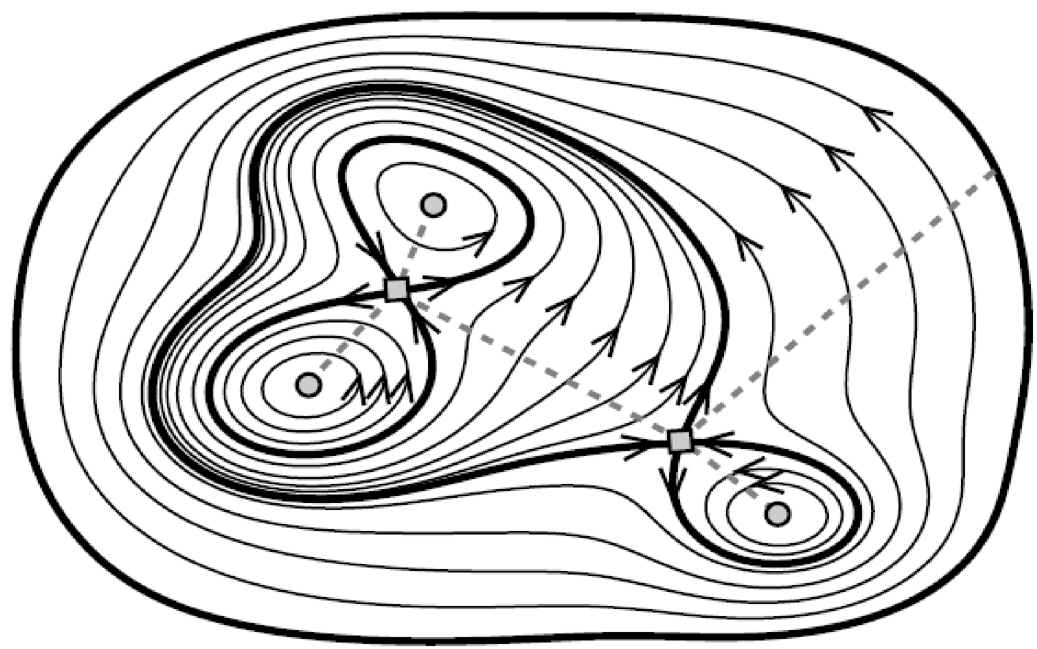
Classic: look for local maxima of $\int \boldsymbol{E} \cdot \boldsymbol{B}$

Partition fluxes 2D:
(Yeates, Hornig 2011b)

$$\boldsymbol{B} = \boldsymbol{\nabla} \times (A\boldsymbol{e}_z)$$

Reconnection rate = magnetic flux through boundaries (spearatrices):

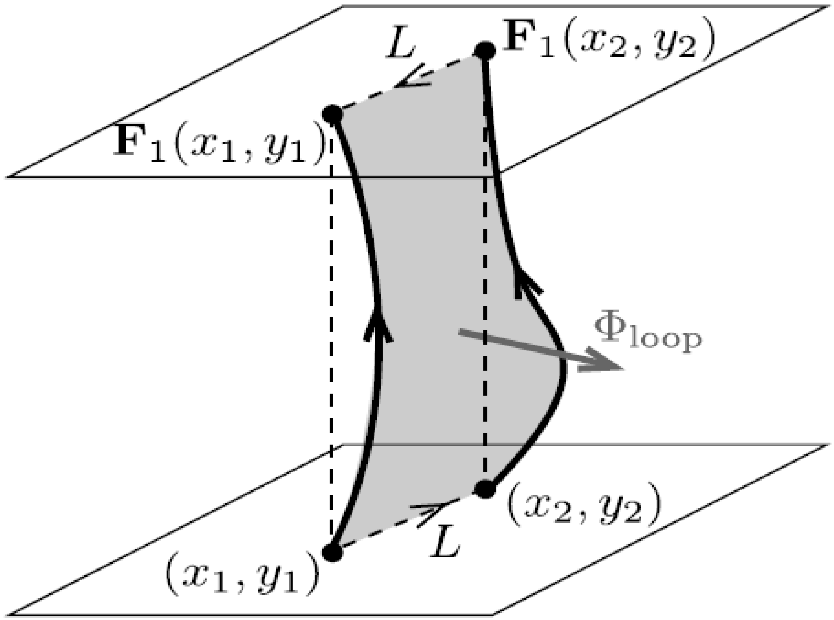
$$\Delta\phi = \sum_i \left| \frac{\mathrm{d}A(\boldsymbol{h}_i)}{\mathrm{d}t} \right|$$



2D Magnetic field.
Thick lines: separatrices.
(Yeates, Hornig 2011b)

Magnetic Reconnection Rate

Partition reconnection rate 3D:
Yeates, Hornig 2011b



Generalized flux function (curly A):

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \boldsymbol{A}\cdot\boldsymbol{B}/B_z \,\mathrm{d}z$$

$$\phi = \int_S \boldsymbol{\nabla}\times\boldsymbol{A}\cdot\,\mathrm{d}\boldsymbol{s} = \int_C \boldsymbol{A}\cdot\,\mathrm{d}\boldsymbol{l}$$

$$\frac{\partial\mathcal{A}}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\mathcal{A} = 0$$

→ invariant in ideal MHD

Fixed points: $\mathbf{F}_1(x_i, y_i) = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$

Reconnection rate:

$$\Delta\phi = \sum_i \left| \frac{\mathrm{d}\mathcal{A}(\boldsymbol{h}_i)}{\mathrm{d}t} \right|$$

Summary

- Braided magnetic fields are observed in the universe.
- Braiding increases stability through the *realizability condition*.
- Turbulent magnetic field decay is restricted by magnetic helicity.
- Knots and links can be represented as braids.
- Fixed point index as additional constraint in relaxation.
- 'Curly A' as measure for the reconnection rate.

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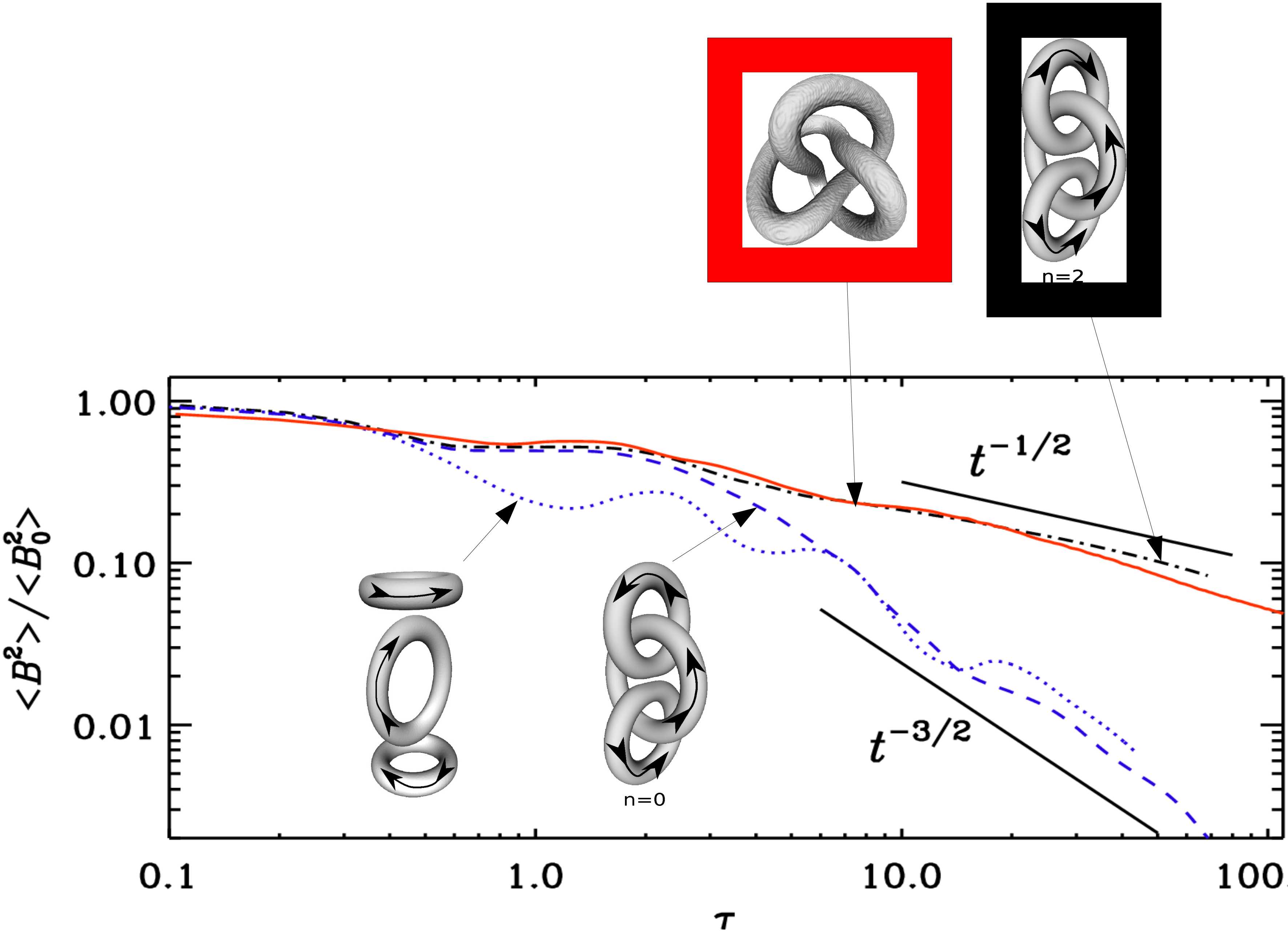
Magnetic energy decay



Simulations

- 256^3 mesh point
- Isothermal compressible gas
- Viscous medium
- Periodic boundaries

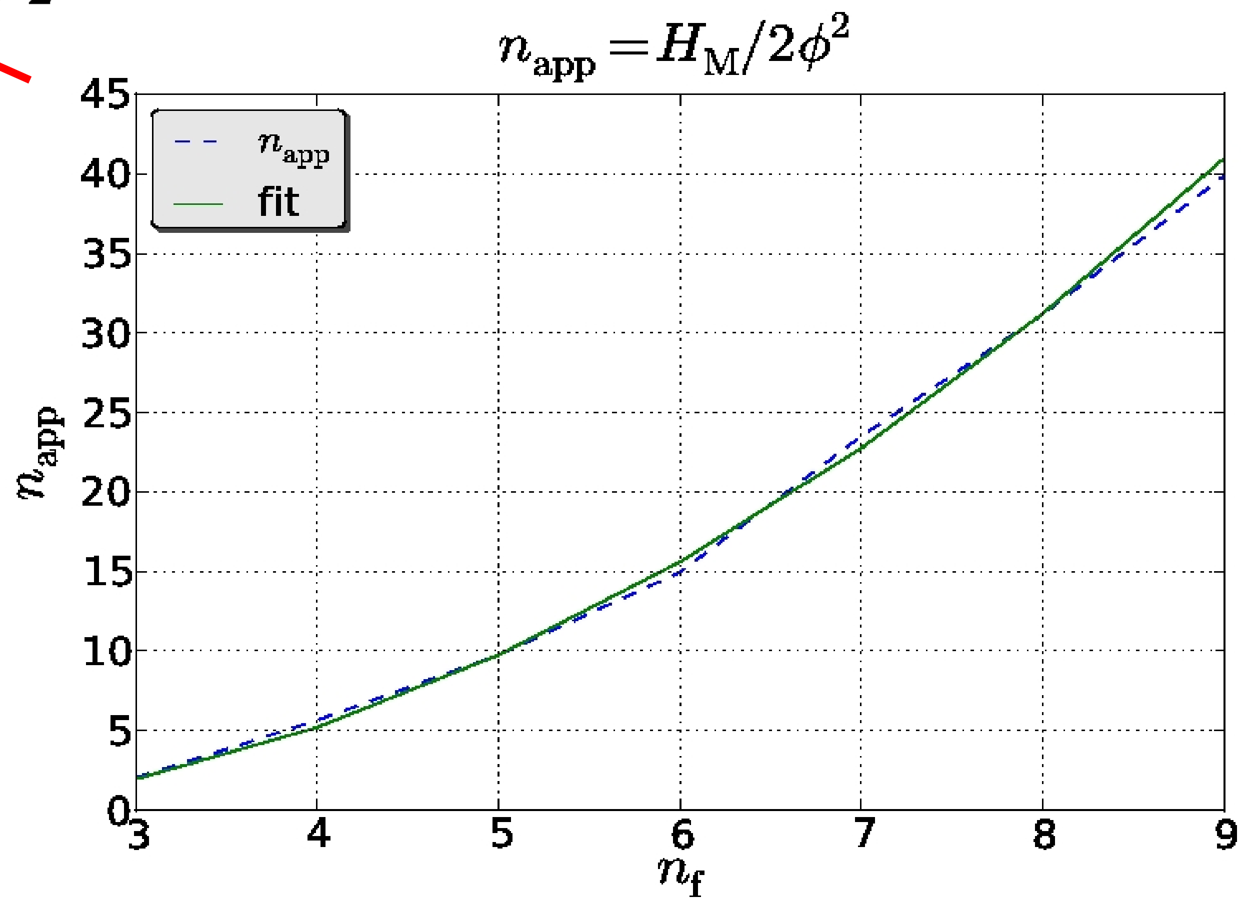
$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} + \eta \nabla^2 \mathbf{A}$$

$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} = -c_{\mathrm{S}}^2 \boldsymbol{\nabla} \ln \rho + \mathbf{J} \times \mathbf{B}/\rho + \mathbf{F}_{\mathrm{visc}}$$

$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{U}$$

N-foil Knots

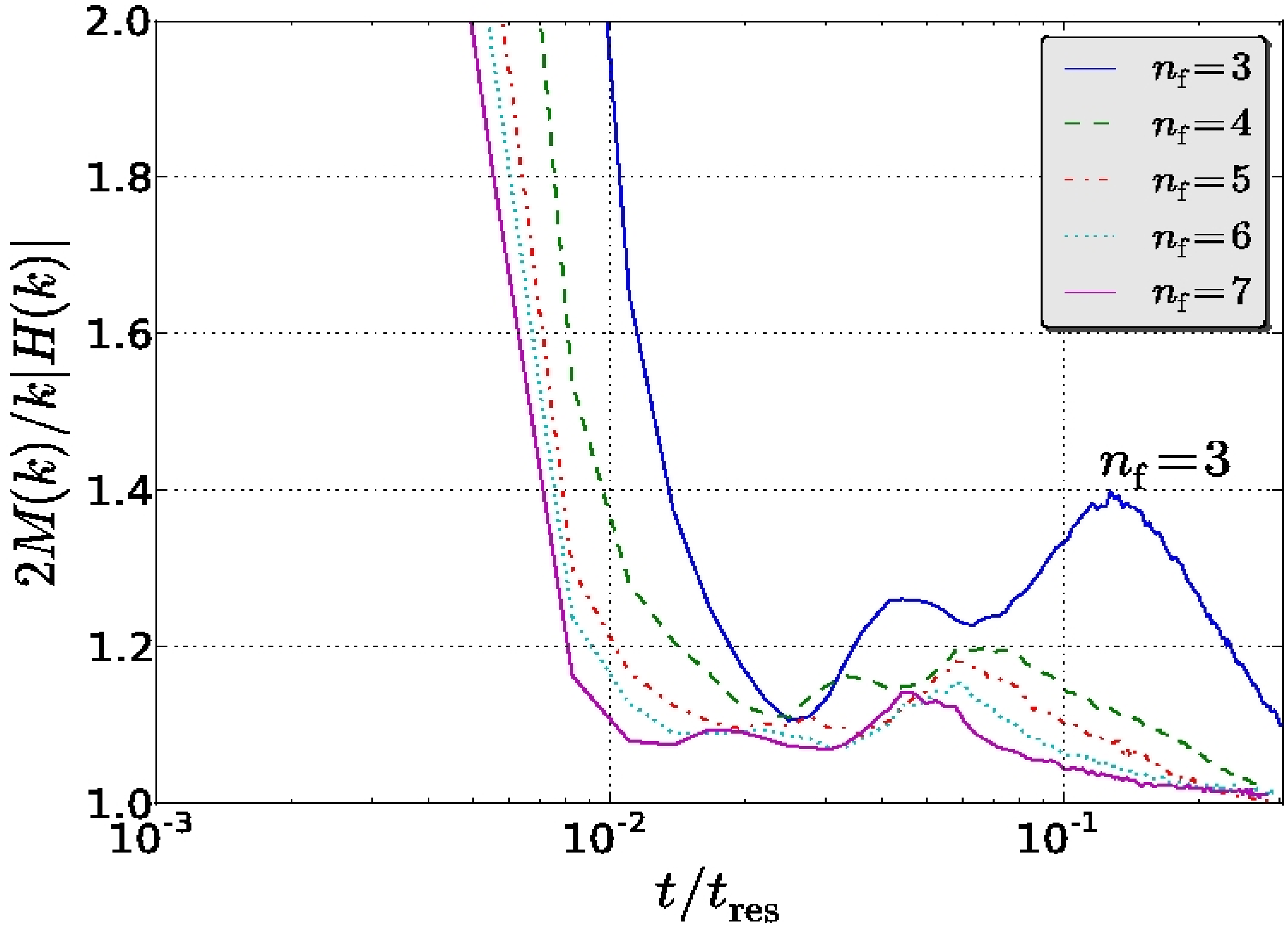
$$H_{\mathrm{M}} = 2n\phi_1\phi_2$$



$$H_{\mathrm{M}} = (n_{\mathrm{f}} - 2)n_{\mathrm{f}}\phi^2/2$$

N-foil Knots

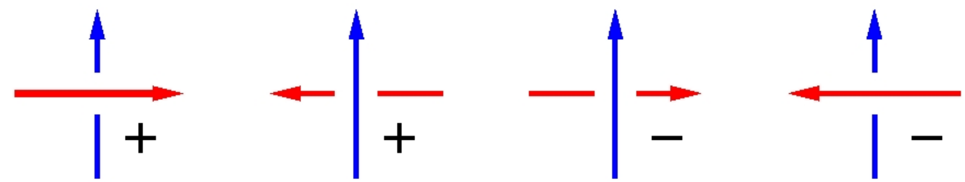
$$2M(k)/(|H(k)|k)$$

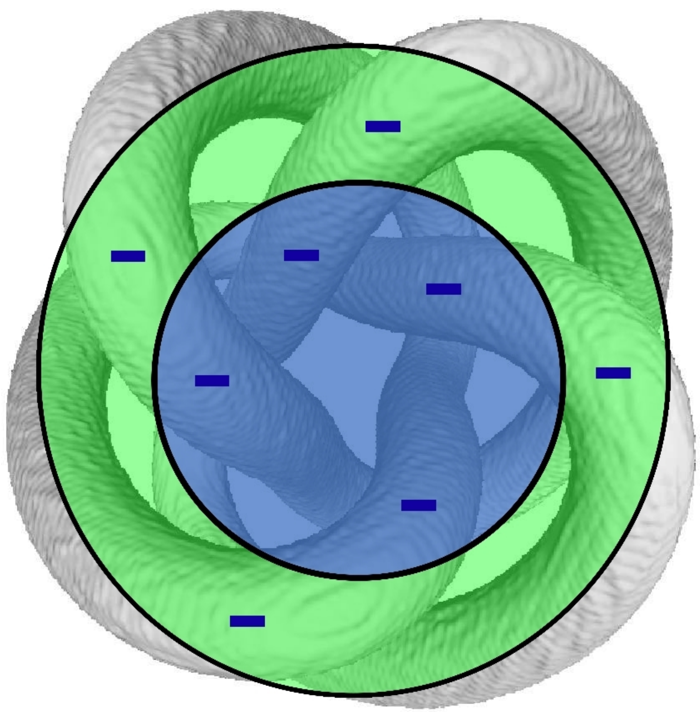


$k = 2$

Realizability condition more important for high $n_{\rm f}$.

Linking Number





Sign of the crossings
for the 4-foil knot

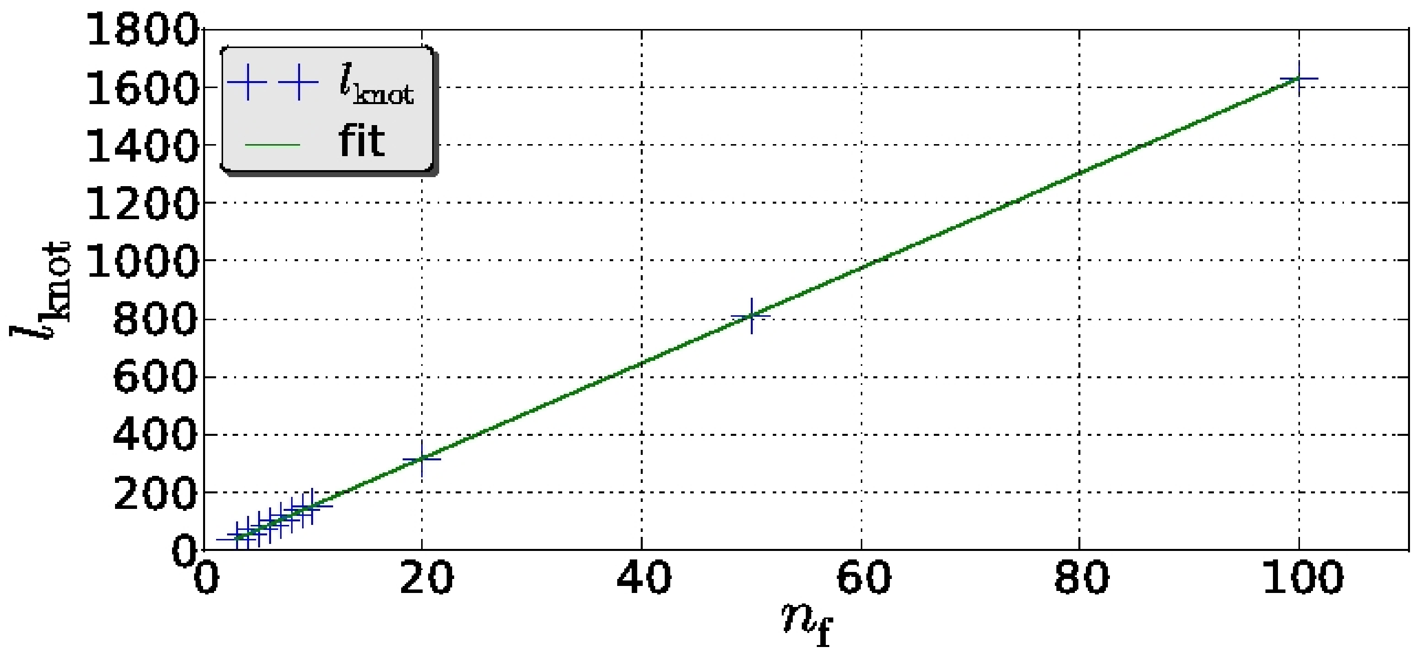
$$n_{\text{linking}} = (n_+ - n_-)/2$$

Number of crossings increases like n_{f}^2

$$H_{\text{M}} \propto n_{\text{linking}}$$

$$H_{\text{M}} \propto n_{\text{f}}^2$$

Helicity vs. Energy



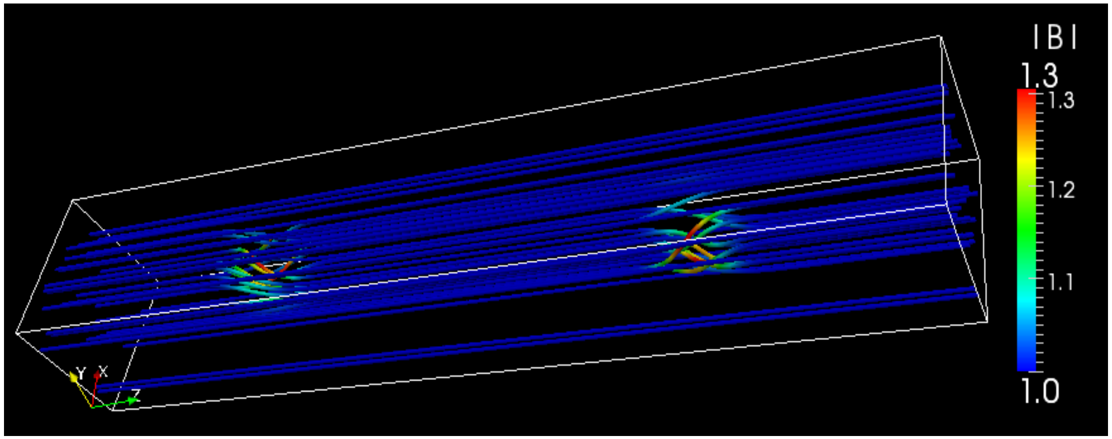
$$E_{\mathrm{M}} \propto l_{\mathrm{knot}} \propto n_{\mathrm{f}}$$

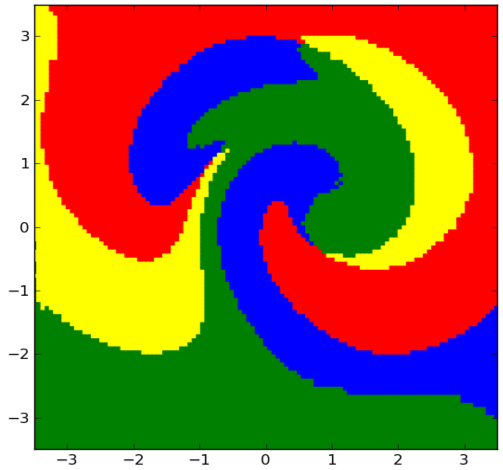
$$H_{\mathrm{M}} \propto n_{\mathrm{f}}^2$$

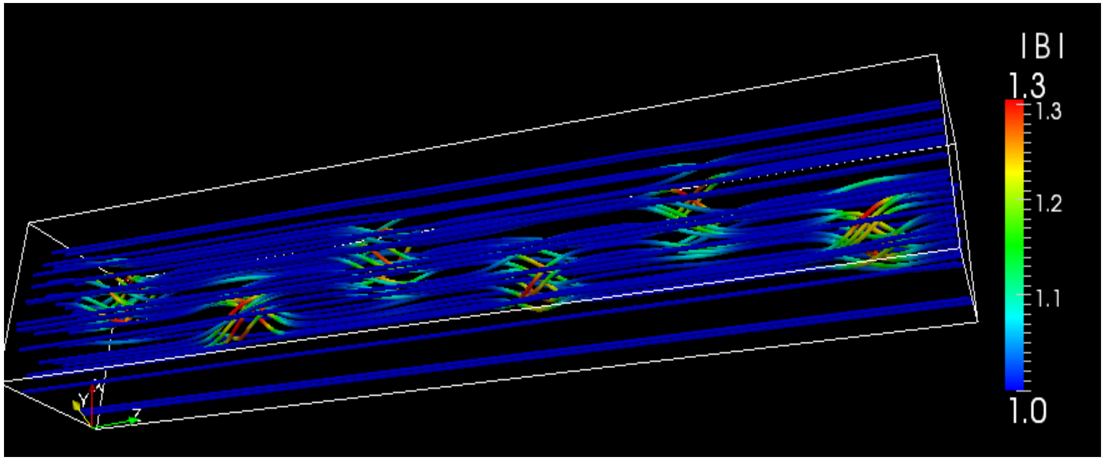
Knot is more strongly packed with increasing .

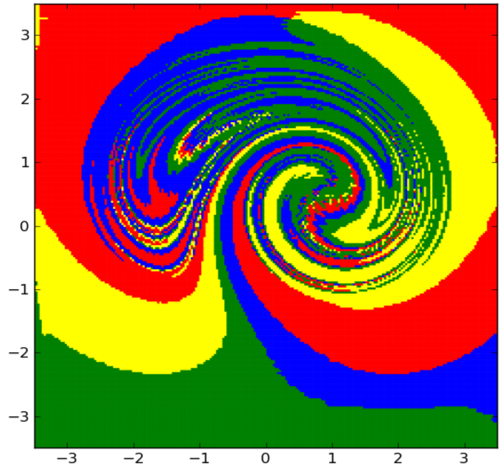
Magnetic energy is closer to its lower limit for high .

Field Line Tracing









Generalized flux function:

$$\mathcal{A}(x,y) = \int_{z=0}^{z=1} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

Reconnection rate:

$$\sum_i \frac{\mathrm{d}\mathcal{A}(\mathbf{x}_i)}{\mathrm{d}t}$$