Topological constraints in magnetic field relaxation

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Abstract.
   Stability and reconnection of magnetic fields play a fundamental role in natural and man-made plasma. In these applications the field's topology determines the stability of the magnetic field. Here I will describe the importance of one topology quantifier, the magnetic helicity, which impedes any free decay of the magnetic energy. Further constraints come from the fixed point index which hinders the field to relax into the Taylor state.

1. Introduction

Examples of topologically intriguing magnetic fields can be found both in nature and in laboratory plasma. During solar eruptions, ejections of hot plasma often appear in the shape of pig tail like loops or sigmoidal structures (Canfield *et al.*, 1999). These shapes are the product of strong magnetic fields which force the charged particles to move along the field lines, which themselves have a loop like shape. In laboratory plasma topologically non-trivial magnetic fields are used to shield the walls of the apparatus from the hot plasma. In order to improve stability, the field is often twisted (Taylor, 1986) and can form even more sophisticated helical shapes, such as those in the Large Helical Device in Toki, Japan (Motojima O. et al., 2005).

During their evolution, astrophysical magnetic fields can break up, reconnect or simply diffuse, which does not only alter their geometry but in some cases also their topology. Two magnetic vector fields are topologically different if one cannot be transformed into the other without magnetic reconnection, i.e. without breaking field lines and connecting them in a different manner. Examples are magnetic flux tubes of the shape of knots and links (Fig. 1–Fig. 3).
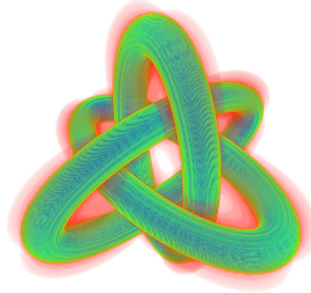
2. Magnetic Helicity

Magnetic helicity $H = \int \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V$, where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ is the magnetic field expressed as its vector potential $\boldsymbol{A}$, is the most widely used measure for qualifying and quantifying the field's topology. It measures the number of mutual linkage of two magnetic field lines (Moffatt, 1969). This popularity arises from both its universal applicability, i.e. it is useful for delimited and diffusive fields, and from its conservation in ideal MHD.

In astrophysics and plasma physics one is often interested in the stability of the field. Magnetic helicity is approximately conserved in such systems, suggesting that helical systems are more stable. Arnold (1974) captured this characteristic in the realizability condition, which expresses
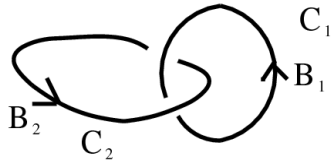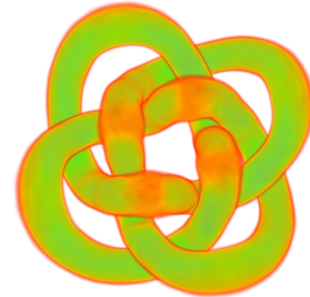
Figure 1. Hopf link consisting of magnetic flux tubes.

Figure 2. Volume rendering of the initial magnetic energy for the Borromean rings configuration.

Figure 3. Volume rendering of the initial magnetic energy for the 8_{18} knot.

a lower limit for the magnetic energy in presence of magnetic helicity

$$E(k) \geq k|H(k)|/(2\mu_0), \tag{1}$$

with the spectral magnetic energy $E(k)$, magnetic helicity $H(k)$, the wave number k and the vacuum permeability μ_0. It should be noted that, although magnetic helicity is generally gauge dependent, the realizability condition is not, which preserves its physical relevance.

Helical field configurations like the Hopf link (see Fig. 1) consist of field lines which are mutually interlocked or self interlinked. Such linkage obviously obstructs the evolution of the field as long as magnetic reconnection is not very strong (Moffatt, 1985). This resistance against free evolution is captured in the realizability condition (1). There are, however, field configurations of interlinked field lines which are not helical, like the Borromean rings (Fig. 2) and the 8_{18} knot (Fig. 3). Are such magnetic links and knots restricted in their evolution because of their linkage, or are they free, because the realizability condition does not apply?

In a resistive and viscous plasma it is easy to simulate the decay of magnetic knots and links (Del Sordo *et al.*, 2010). Three initial magnetic field configurations are considered, each comprising three flux rings (Fig. 4). The first and second configurations are linked, the first of which is helical. The third un-linked setup servers for comparison. Initially all three setups show a very similar decay in magnetic energy (Fig. 5). After about 10 Alfvenic times, however, they decay distinctively. For the helical setup we observe a rather slow decay in magnetic energy, which for late times, behaves approximately as a power law of $t^{-1/3}$. For the two non-helical configurations this power law is steeper and of the order of $t^{-3/2}$. Unlike it might be assumed, the actual linking of the rings does not affect the dynamics considerably. Magnetic helicity, on the other hand, restricts the relaxation such that magnetic energy decays only slowly. The two non-helical cases decay equally fast and the significance of the realizability condition is confirmed.

3. Beyond magnetic helicity
Other than the magnetic helicity there exist third and forth order topological invariants which are non-zero for the Borromean rings (Ruzmaikin & Akhmetiev, 1994) . Whether such invariants affect the field's evolution is tested in comparison with the triple rings (Fig. 6) (Candelaresi & Brandenburg, 2011). Reconnection causes the Borromean rings to reshape into two twisted tori (Fig. 7). Those tori are helical, and so the realizability condition has an effect locally, which can be observed in the slowed down decay of the magnetic energy (Fig. 6). Any higher order topological invariant is not needed to explain the intermediate decay, but cannot be excluded either.
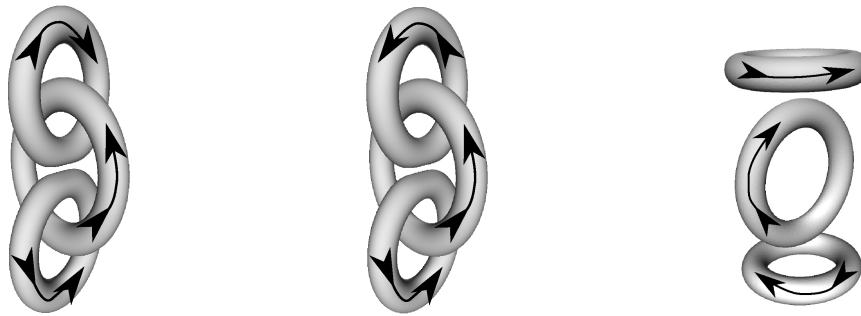
Figure 4. Iso surfaces for the initial magnetic field for the triple ring configurations. The left configuration is helical, while the other two are non-helical. Taken from (Del Sordo *et al.*, 2010).
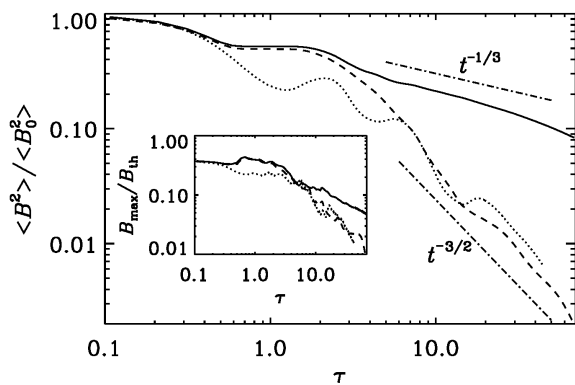
Figure 5. Normalized magnetic energy in time for the helical triple rings (solid line), the non-helical interlocked rings (dashed line) and un-linked rings (dotted line). The linking of field lines is not able to slow down the decay of magnetic energy. On the other hand, the presence of magnetic helicity poses a constraint in the field's relaxation. Taken from (Del Sordo *et al.*, 2010).
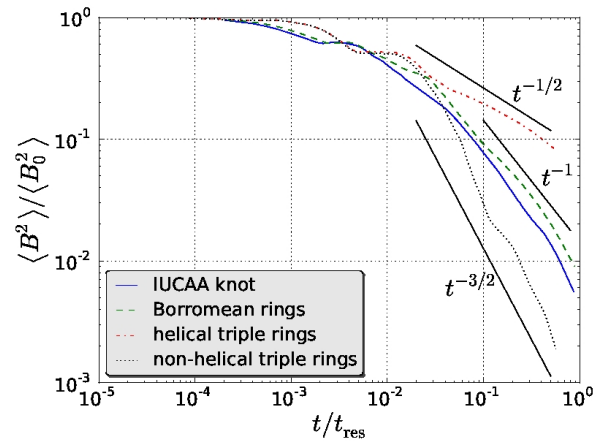
Figure 6. Comparison of the magnetic energy evolution for the interlocked triple rings, the Borromean rings and the IUCAA knot (8_{18} knot). The non-helical Borrmean rings and the IUCAA knot show an intermediate decay behavior, of which the first one can be explained by the occurrence of separated helical structures. Taken from (Candelaresi & Brandenburg, 2011).

Perhaps the most successful application of topological invariants other than the magnetic helicity is the fixed point index (Brown, 1971). Yeates *et al.* (2010) showed that its conservation imposes an additional restriction in the field's evolution such that the relaxed state reaches higher energies than proposed by Taylor (1974). Any application of the fixed point index on particularly non-helical knots and links in the form of braids would give considerable new insights in their relaxation dynamics.

4. Conclusions

We have shown that magnetic helicity, rather than the topology, is the most important parameter in determining the evolution of magnetic links and knots. Nevertheless, a non-helical initial field can spontaneously transform into separate helical fields which are restricted in their evolution (see Borromean rings). Combining the fixed point index with various topologically non-trivial
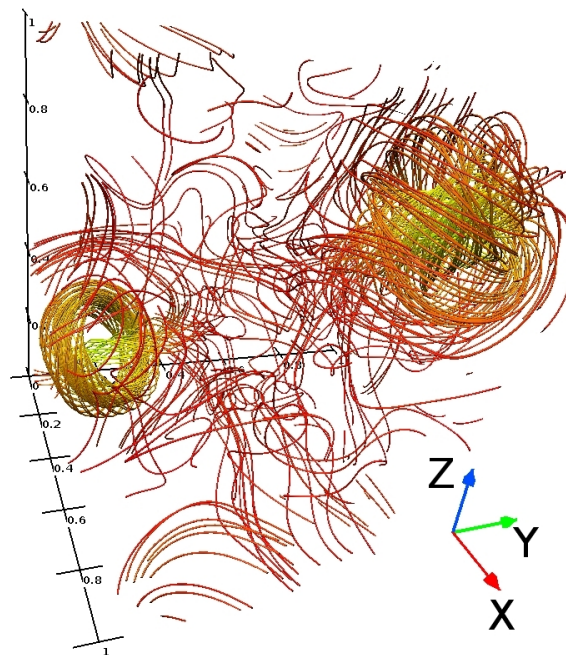
Figure 7. Magnetic streamlines of the Borromean rings configuration after a decay over 78 Alfvenic times. The colors represent the strength of the magnetic field. Two helical structures appear which are spatially separated. Their helicity content impedes any fast decay of magnetic energy. Taken from (Candelaresi & Brandenburg, 2011).

knots and links would enhance our understanding of topological constraints and magnetic reconnection.

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